Platonic Polyhedra and How to Construct Them

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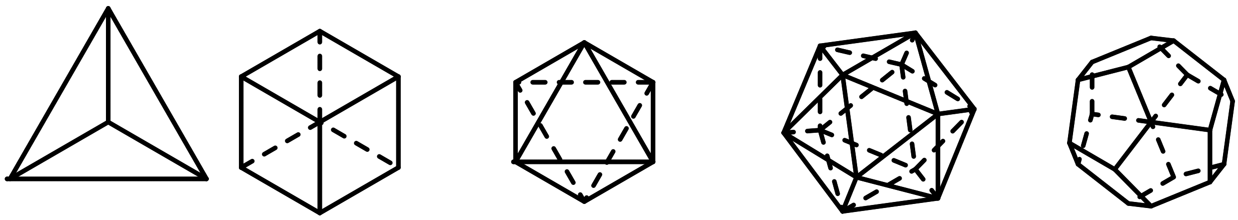

The platonic polyhedra (or platonic solids) are convex regular polyhedra that have identical regular polygons as faces. They are characterized by two numbers – the number of sides of each face and the number of faces that meet at a corner (or vertex) of the polyhedron. If three triangular faces meet at a corner, we have a **tetrahedron** (four faces). If four triangular faces meet at a corner, we have an **octahedron** (eight faces). If five triangular faces meet at a corner, we have a **icosahedron** (twenty faces). Six triangular faces meeting at a corner do not produce any surface curvature and hence, cannot close in space to produce a polyhedron. More than six is, of course, not possible. For square faces, the only possibility is three of them meeting at a corner producing a **cube** (or **hexahedron**). Four such faces do not produce surface curvature and any more than four is not possible. Similarly, for pentagonal faces the only possibility is three of them meeting at a corner producing a **dodecahedron** (twelve faces). Hexagonal faces meeting at a corner can produce no polyhedra as three of them meeting at a corner produce no curvature and any higher number is not possible. Polygons with more than six sides cannot meet at a corner and hence, the above five are the only possibilities for convex regular polyhedra.

In the following I shall first construct the polyhedra with triangular faces and then the dodecahedron. I shall leave out the cube as it is trivial to construct. However, the other four polyhedra can actually be visualized as inscribed inside a cube. Such visualization allows the assignment of coordinates to the corners in a rectangular coordinate system. This is helpful in constructing various projections of the polyhedra (maybe in a computer program).

1 Tetrahedron

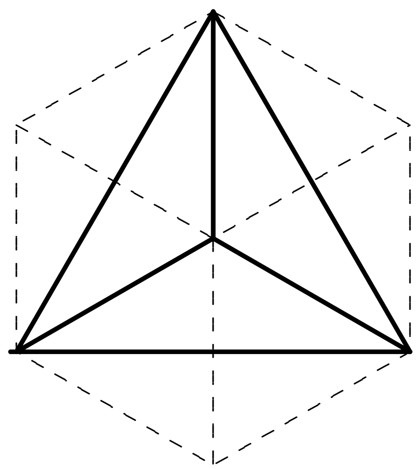

Figure 1: Tetrahedron in a cube. Isotropic projection.

The tetrahedron is the easiest to construct. Figure 1 shows how to construct a tetrahedron by drawing six of the face diagonals of a cube. If the center of the cube is the origin and the length of a side of the cube is d, then the coordinates of the four corners are:

$$(-d/2, -d/2, d/2) \quad (-d/2, d/2, -d/2) \quad (d/2, -d/2, -d/2) \quad (d/2, d/2, d/2) \tag{1}$$

2 Octahedron

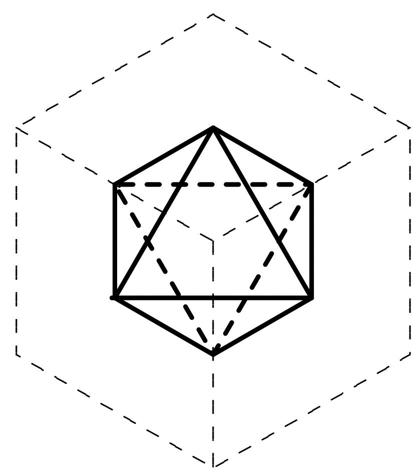

Figure 2: Octahedron in a cube. Isotropic projection.

Figure 2 shows how to draw an octahedron by joining face centers of a cube. If the center of the cube is the origin and the length of a side of the cube is d, then the coordinates of the six corners are:

$$(-d/2, 0, 0) \quad (d/2, 0, 0) \quad (0, -d/2, 0) \quad (0, d/2, 0) \quad (0, 0, -d/2) \quad (0, 0, d/2) \tag{2}$$

3 Icosahedron

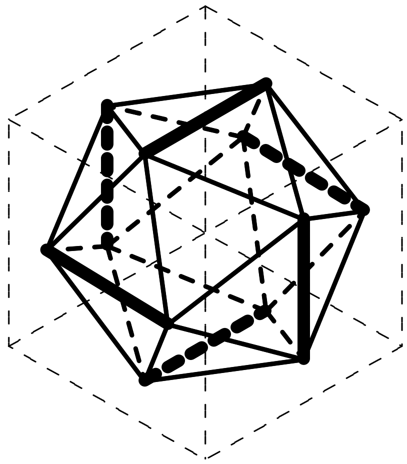

Figure 3: Icosahedron in a cube. Isotropic projection.

The icosahedron inscribed in a cube is more difficult to visualize. Figure 3 helps to some extent. Note the sides shown in thick lines (solid or dashed). They are drawn to be contained in the six cube faces. They are parallel in pairs on opposite faces. Every pair is perpendicular to the other pairs. Note that the ends of these lines define all the twelve corners of the icosahedron. Hence, joining these corners appropriately forms the icosahedron.

An alternate view of the icosahedron in a cube is shown in figure 4. This is the view as seen perpendicular to one of the cube faces (say, top face). Note that there are no hidden lines in this view. This is because the hidden lines are all exactly under the visible lines. So there are eight faces on the top and eight more below. The remaining four faces are in edge view on the left and right sides.

In the following, I shall show that this fitting of the icosahedron in a cube is indeed possible if the length of a side of the icosahedron has a specific relation to the length of a side of the cube. Such a fitting allows the assignment of coordinates to the corners in a rectangular coordinate system. This is helpful in drawing various projections of the icosahedron (maybe in a computer program).

3.1 Inscribing a Icosahedron in a Cube

Consider the fact that all sides of the icosahedron are equal and the faces are equilateral triangles. Five of these triangles meet at each corner. Using this information alone, it can be shown that a icosahedron can be inscribed in a cube. One can demonstrate it as follows.

Let us imagine a closed planar maximal loop on the surface of the icosahedron. Consider one side of the icosahedron to be the first side of this loop. Then, to keep the loop planar and maximal, the second side has to be the angle bisector of an adjacent face. The third side will be the angle bisector of the next face. The fourth side will be another side of the

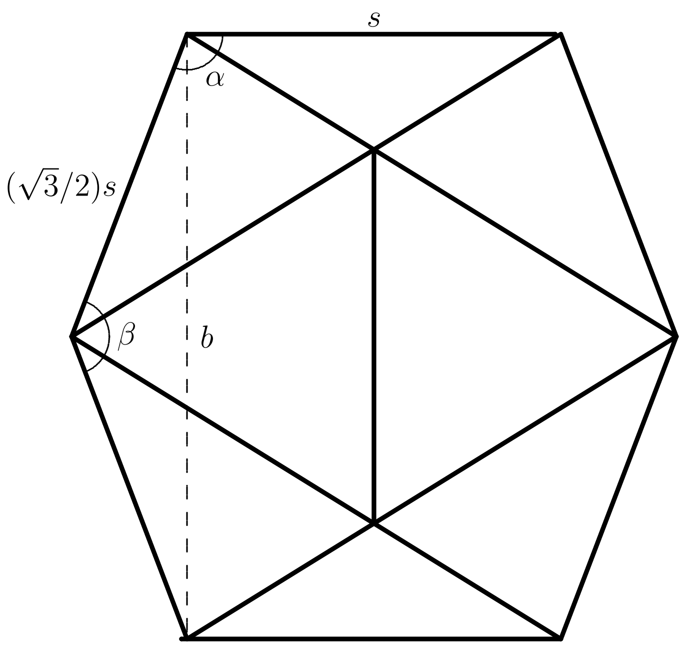

Figure 4: Icosahedron in a cube. Orthographic projection (top view).

icosahedron. The next two sides of the loop will once again be angle bisectors of faces. It turns out that the loop closes with just these six sides. In the following, I shall prove that this does actually happen. One such possible loop would be the outer boundary of the icosahedron in the view shown in figure 4. As a result, the sides of the loop have the lengths as shown in the figure. The length of a side of the icosahedron is taken to be s. Hence, the angle bisector of a triangular face is $\sqrt{3}s/2$.

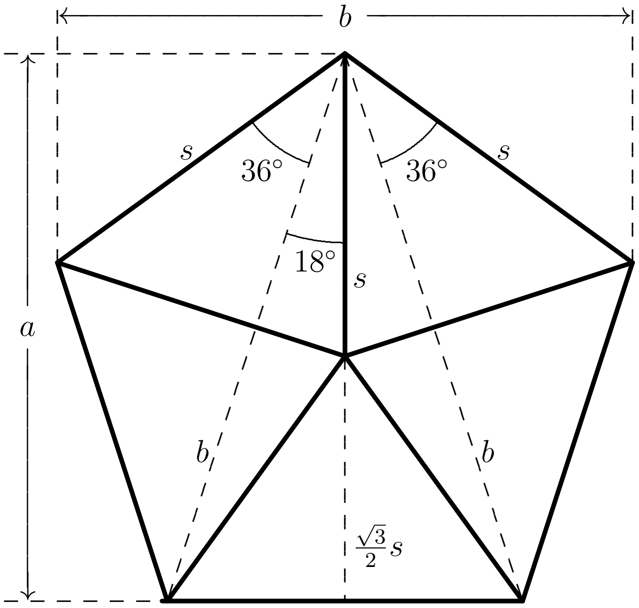

Figure 5: Five triangles meeting at a corner. Pentagon in true shape.

To find b, α and β as shown in the figure, we construct the view shown in figure 5. It shows the five triangles meeting at a corner. The pentagonal shape is in true shape. From the construction it can be seen that

$$b = 2s\sin 54^\circ = 1.618034s. \tag{3}$$

The length a, as shown in the figure is as follows.

$$a = b\cos 18^\circ = 2s\sin 54^\circ \cos 18^\circ = 1.5388418s. \tag{4}$$

Now consider the triangle in a plane perpendicular to the page in figure 5 that has one side of length s, another running down the middle of the opposite face (an angle bisector and hence of length $\sqrt{3}s/2$) and the third of length a. Notice that the angle opposite to the side of length a is the angle α of figure 4. Hence, using the cosine formula we get

$$s^2 + 3s^2/4 - a^2 = \sqrt{3}s^2 \cos\alpha \tag{5}$$

This gives

$$\alpha = 110.90516^\circ \tag{6}$$

Figure 4 shows that

$$b = 2(\sqrt{3}/2)s\sin(\beta/2), \tag{7}$$

and hence,

$$\beta = 138.18969^\circ. \tag{8}$$

Now it is evident that

$$4\alpha + 2\beta = 720^\circ. \tag{9}$$

This is the sum of the interior angles of a hexagon. Hence, it is proved that the planar maximal loop that is the outer boundary of the icosahedron projection in figure 4 is indeed a hexagon. Wrapping around sets of one side and two faces to complete a polygon could produce a triangle[1], or a hexagon, or a nonagon or any other polygon with number of sides that is a multiple of three. The fact that our computed values of α and β actually satisfy equation 9, proves that the boundary polygon is a hexagon. To show that the fitting of the icosahedron in a cube is correct, we also need to show the equality of the height and the width of the dashed line square in figure 4. This is easily done by noticing that the width is

$$w = (\sqrt{3}/2)s\cos(\beta/2) + s + (\sqrt{3}/2)s\cos(\beta/2), \tag{10}$$

which turns out to be the same as b, the height.

Using the computed values of α and β and the values of a and b in terms of the side length s, it is straightforward to set up coordinates for the corners of a icosahedron. However, it is even simpler to set up the coordinates using the setting inside a cube as shown below.

[1]Note that this is the case of the tetrahedron

3.2 Using the Fitting in a Cube to Draw the Icosahedron

Once we accept the setting of the icosahedron in a cube, the computations become much simpler. As the two sides of the square in figure 4 must be the same, we get

$$2s(\sqrt{3}/2)\cos(\beta/2) + s = b \tag{11}$$

$$2s(\sqrt{3}/2)\sin(\beta/2) = b \tag{12}$$

Then, eliminating β from the equation, we get

$$1 = \cos^2(\beta/2) + \sin^2(\beta/2) = \left(\frac{b-s}{\sqrt{3}s}\right)^2 + \left(\frac{b}{\sqrt{3}s}\right)^2 \tag{13}$$

This gives

$$s = \left(\frac{\sqrt{5}-1}{2}\right) b. \tag{14}$$

Then, from equation 12, we get

$$\sin(\beta/2) = \left(\frac{2}{(\sqrt{5}-1)\sqrt{3}}\right). \tag{15}$$

Hence,

$$\beta = 138.18969^\circ. \tag{16}$$

Now that s is known in terms of b the side of the cube, one may draw the six sides of the icosahedron on the six faces of the cube as shown in figure 3 with bold lines. The ends of these six sides define all twelve corners of the icosahedron. Hence, drawing the icosahedron amounts to joining these corners appropriately. In a coordinate system with the origin at the center of the cube, the corners of the icosahedron may be represented by the following points.

$$\begin{array}{llll} (-s/2,-b/2,0) & (s/2,-b/2,0) & (s/2,b/2,0) & (-s/2,b/2,0) \\ (-b/2,0,-s/2) & (b/2,0,-s/2) & (b/2,0,s/2) & (-b/2,0,s/2) \\ (0,-b/2,-s/2) & (0,b/2,-s/2) & (0,b/2,s/2) & (0,-b/2,s/2) \end{array} \tag{17}$$

4 Dodecahedron

Unlike the three polyhedra discussed so far, the dodecahedron has pentagonal faces. Three such pentagons meet at each corner. Nonetheless, the dodecahedron can also be inscribed in a cube (figure 6). Note the sides shown in thick lines (solid or dashed). They are drawn to be contained in the six cube faces. They are parallel in pairs on opposite faces. Every pair is perpendicular to the other pairs. Note that the ends of these lines define twelve corners of the dodecahedron. So far, this construction is similar to the construction of the icosahedron. But the dodecahedron has eight more corners. These corners are not on the cube faces. However, they are the corners of another cube that is smaller than the original.

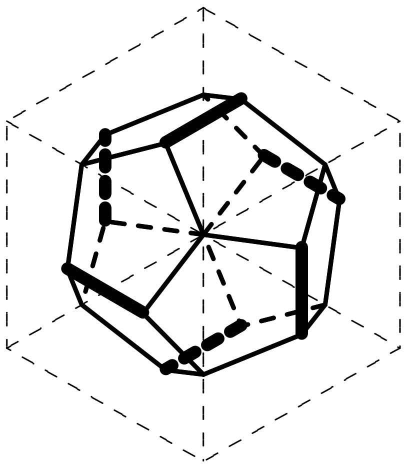

Figure 6: Dodecahedron in a cube. Isotropic projection.

Joining these twenty corners appropriately forms the dodecahedron. The length of the sides of the dodecahedron and the sides of the inner cube in relation to the original cube will be done in the following.

An alternate view of the dodecahedron in a cube is shown in figure 7. This is the view as seen perpendicular to one of the cube faces (say, top face). Note that there are no hidden lines in this view. This is because the hidden lines are all exactly under the visible lines. So there are four faces on the top and four more below. The remaining four faces are in edge view on the left and right sides.

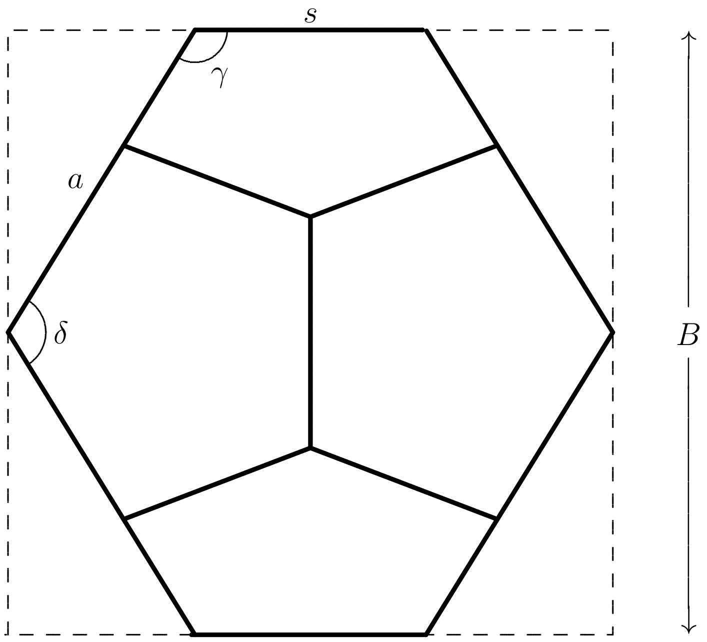

Figure 7: Dodecahedron in a cube. Orthographic projection (top view).

In the following, I shall show that this fitting of the dodecahedron in a cube is indeed possible if the length of a side of the dodecahedron has a specific relation to the length of a side of the cube. Such a fitting allows the assignment of coordinates to the corners in a rectangular coordinate system. This is helpful in drawing various projections of the dodecahedron (maybe in a computer program).

4.1 Inscribing a Dodecahedron in a Cube

As in the case of the icosahedron, let us imagine a closed planar maximal loop on the surface of the dodecahedron. Consider one side of the dodecahedron to be the first side of this loop. Then, to keep the loop planar and maximal, the second side has to be the angle bisector of an adjacent face. The third side will be the angle bisector of the next face. The fourth side will be another side of the dodecahedron. The next two sides of the loop will once again be angle bisectors of faces. It turns out, once again, that the loop closes with just these six sides. In the following, I shall prove that this does actually happen. One such possible loop would be the outer boundary of the dodecahedron in the view shown in figure 7. As a result, the sides of the loop have the lengths as shown in the figure. The length of a side of the dodecahedron is taken to be s. The angle bisector of a pentagonal face is a which has the same relation to s as in figure 5. In the following, I shall find the cube side B in terms of s. I shall also find the angles γ and δ.

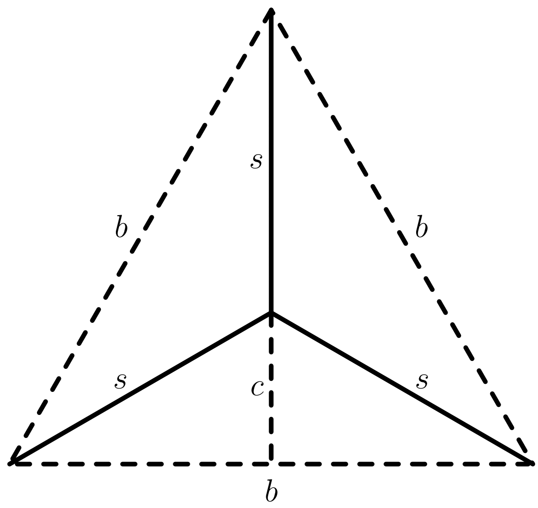

Figure 8: Dodecahedron corner.

To find γ, let us focus on one corner of the dodecahedron as shown in figure 8. In this figure, the equilateral triangle in dashed lines is in true shape. All other lines are projected in the plane of this triangle. The relationship of b (the side of the equilateral triangle) to s is the same as in figure 5. Hence it is given by equation 3. As the angle between the sides of a pentagon is 108°, it can be seen that the length c of the right bisector shown is

$$c = s \cos 54^\circ = 0.5877852s. \tag{18}$$

Now consider the triangle in a plane perpendicular to the page that has one side of length c, one of length s and one of length $\sqrt{3}b/2$ (the length of an angle bisector of the equilateral triangle). It can be seen that the angle γ is the angle between the sides of length s and c. Hence, using equations 3 and 18 and the cosine formula,

$$\cos\gamma = \frac{c^2 + s^2 - 3b^2/4}{2cs} = -0.5257312. \tag{19}$$

So,

$$\gamma = 121.71748°. \tag{20}$$

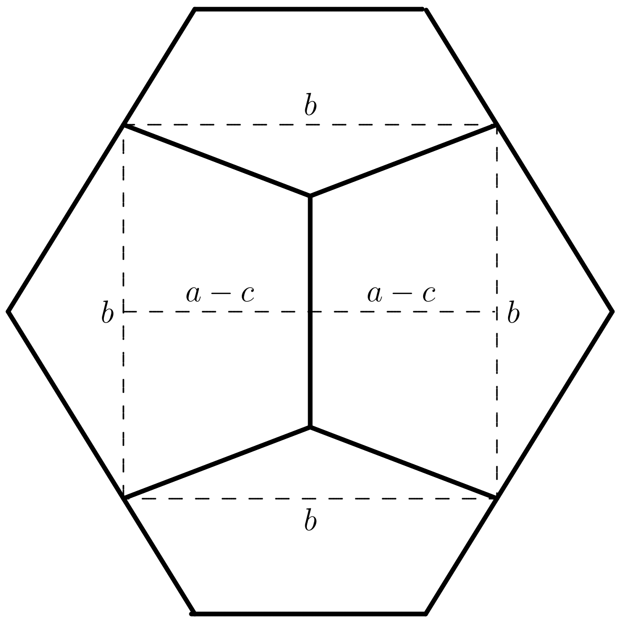

Figure 9: Four Faces of a Dodecahedron.

Next, consider the four pentagonal faces in the view shown in figure 7. Figure 9 shows just these four faces to clarify the computation of the angle δ. The length b has already been computed in equation 3. Here we identify a square of side b as shown by dashed lines. This square is in true shape. The dashed lines of length $a - c$ are on the pentagonal faces and they are shown in projection. The angle between these two lines can be seen to be the angle δ. Hence, using the triangle with these two lines as two sides and the third side of length b, it is seen that

$$\sin\delta/2 = \frac{b/2}{a - c} = 0.8506508, \tag{21}$$

where we use equations 3, 4 and 18 to find a, b and c in terms of s. This gives

$$\delta = 116.56505. \tag{22}$$

Now the fact that

$$4\gamma + 2\delta = 720°, \tag{23}$$

once again shows that the maximal planar loop is a hexagon. Also, we can now see that the sides of the cube in figure 7 are indeed equal. One of the sides is

$$B = 2a \sin \delta/2 = 2.618034s. \tag{24}$$

The other side is

$$W = 2a \cos \delta/2 + s = 2.618034s. \tag{25}$$

4.2 Using the Fitting in a Cube to Draw the Dodecahedron

Now that we have seen the fitting of the dodecahedron in a cube, the coordinates of the twelve corners on the cube faces are given by

$$\begin{array}{llll} (-s/2, -B/2, 0) & (s/2, -B/2, 0) & (s/2, B/2, 0) & (-s/2, B/2, 0) \\ (-B/2, 0, -s/2) & (B/2, 0, -s/2) & (B/2, 0, s/2) & (-B/2, 0, s/2) \\ (0, -B/2, -s/2) & (0, B/2, -s/2) & (0, B/2, s/2) & (0, -B/2, s/2) \end{array} \tag{26}$$

where the relationship of B and s is given by equation 24. Hence,

$$s = 0.381966B. \tag{27}$$

The other eight corners are at the corners of a cube of side b as seen in figure 9. So, the coordinates of these are

$$\begin{array}{cccc} (-b/2, -b/2, -b/2) & (-b/2, -b/2, b/2) & (-b/2, b/2, -b/2) & (-b/2, b/2, b/2) \\ (b/2, -b/2, -b/2) & (b/2, -b/2, b/2) & (b/2, b/2, -b/2) & (b/2, b/2, b/2) \end{array} \tag{28}$$

where the relationship of b and s is given by equation 3. Hence,

$$b = 1.618034s = 0.6180339B. \tag{29}$$

5 Regular Polyhedra in Physics and Chemistry

The shapes discussed here often show up in physics and chemistry. Here are some examples.

5.1 SP^3 Hybridization for the Carbon Atom

In organic compounds, saturated carbon atoms show a tetrahedral symmetry. The methane molecule shows the perfect case. It has one carbon atom attached to four hydrogen atoms in a perfectly symmetric form. Hence, the carbon is at the center of a tetrahedron and the four hydrogen atoms at the corners. Using the fitting of a tetrahedron in a cube (figure 1), it is straightforward to compute the angles between bonds. If the sides of the cube are of length d and the origin of the coordinate system is at the center of the cube, then the two position vectors of the two endpoints of a face diagonal are

$$\vec{\mathbf{A}} = (-d/2, -d/2, d/2) \tag{30}$$

$$\vec{\mathbf{B}} = (d/2, d/2, d/2) \tag{31}$$

The angle θ between these two vectors is the bond angle. The cosine of θ is given by

$$\cos\theta = \frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{|\vec{\mathbf{A}}||\vec{\mathbf{B}}|} = -1/3. \tag{32}$$

Hence,

$$\theta = 109.47^\circ \tag{33}$$

This value of the angle is verified by experiment.

5.2 Face Centered Cubic Crystals

In solid state physics the face-centered cubic crystal structure is very well-known. It is one of the two close-packed structures that crystals of metals display. This structure has an atom at each corner and each face-center of a cubic lattice. We have already seen that the six face centers of a cube define an octahedron. It can also be seen that an atom at a corner of the cube has three face-center atoms as nearest neighbors and they together form a tetrahedron.

5.3 The Buckyball or Buckminsterfullerene

In 1985 a new and unusual form of carbon was synthesized. It had been predicted a couple of decades before that. It was called **buckminsterfullerene** or buckyball for short. The name is a reference to Buckminster Fuller, a pioneer in geodesic dome architecture. This form of carbon is a collection of sixty carbon atoms placed at the corners of an almost spherical structure called the **truncated icosahedron**. This structure is a simple derivative of the usual icosahedron. Consider slicing off (truncating) the twelve corners of the icosahedron. This replaces each corner by a new pentagonal face and five new corners. Hence, it has sixty corners for the sixty carbon atoms. It also has twelve pentagonal faces. The original twenty triangular faces convert to hexagonal faces. It is to be noted that the most popular pattern of soccer ball design in decades has been the truncated icosahedron.

6 The Truncation Connection

In the last section, I described the truncated icosahedron. Let us now study the truncation process more carefully. It is to slice off corners in a symmetric fashion such that the corners of the icosahedron are replaced by planar faces of regular pentagonal shape. In the process, the original triangular faces will have their corners clipped, so that they become hexagons. The truncation could be done just enough such that the hexagons are regular hexagons. However, if the truncation is deeper than that, the original sides from the triangular faces become shorter. In fact, with deep enough truncation, one could make the original sides disappear such that the hexagonal faces become triangular again. With even deeper truncation these triangular faces can reduce down to points, leaving only the twelve pentagonal faces. This, of course, is a dodecahedron! What might be even more surprising is that a similar truncation process of the corners of a dodecahedron converts it to a icosahedron! Hence, the dodecahedron and the icosahedron are called **dual polyhedra**.

Similarly, the cube and the octahedron are also a dual pair. The tetrahedron can be seen to be **self-dual**. Truncating its corners produces another tetrahedron.