Solutions
Chapter 7

Problem 7.6

$$\mathcal{Z} = \sum_s e^{-(E(s)-\mu N(s))/kT}, \tag{1}$$

and by definition,

$$\overline{N} = \frac{1}{\mathcal{Z}} \sum_s N(s) e^{-(E(s)-\mu N(s))/kT} \tag{2}$$

$$\frac{\partial \mathcal{Z}}{\partial \mu} = \sum_s (N(s)/kT) e^{-(E(s)-\mu N(s))/kT} = \frac{1}{kT} \sum_s N(s) e^{-(E(s)-\mu N(s))/kT}. \tag{3}$$

Hence,

$$\overline{N} = \frac{kT}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \mu}. \tag{4}$$

By definition,

$$\overline{N^2} = \frac{1}{\mathcal{Z}} \sum_s N^2(s) e^{-(E(s)-\mu N(s))/kT}, \tag{5}$$

and,

$$\frac{\partial^2 \mathcal{Z}}{\partial \mu^2} = \sum_s (N(s)/kT)^2 e^{-(E(s)-\mu N(s))/kT} = \left(\frac{1}{kT}\right)^2 \sum_s N^2(s) e^{-(E(s)-\mu N(s))/kT}. \tag{6}$$

Hence,

$$\overline{N^2} = \frac{(kT)^2}{\mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial \mu^2}. \tag{7}$$

But,

$$\frac{\partial \overline{N}}{\partial \mu} = kT \left(\frac{1}{\mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial \mu^2} - \frac{1}{\mathcal{Z}^2} \left(\frac{\partial \mathcal{Z}}{\partial \mu} \right)^2 \right). \tag{8}$$

So,

$$kT \frac{\partial \overline{N}}{\partial \mu} = \left(\frac{(kT)^2}{\mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial \mu^2} - \frac{(kT)^2}{\mathcal{Z}^2} \left(\frac{\partial \mathcal{Z}}{\partial \mu} \right)^2 \right) = \overline{N^2} - (\overline{N})^2 \equiv \sigma_N^2. \tag{9}$$

Hence,

$$\sigma_N = \sqrt{kT \frac{\partial \overline{N}}{\partial \mu}}. \tag{10}$$

Problem 7.12

Probability of A being *un*occupied for $\epsilon_A = \mu - x$ is,

$$1 - \overline{n}_A = 1 - \frac{1}{e^{-x/kT}+1} = \frac{e^{-x/kT}+1}{e^{-x/kT}+1} - \frac{1}{e^{-x/kT}+1} = \frac{e^{-x/kT}}{e^{-x/kT}+1} = \frac{1}{e^{x/kT}+1}. \tag{11}$$

This is the same as the probability of B being occupied for $\epsilon_B = \mu + x$:

$$\overline{n}_B = \frac{1}{e^{x/kT}+1}. \tag{12}$$

Problem 7.14

Of the three, Bose-Einstein gives the highest and Fermi-Dirac gives the lowest. So, for a maximum of 1% difference, these two should differ by 0.01 at most. Hence,

$$2\frac{\overline{n}_{BE} - \overline{n}_{FD}}{\overline{n}_{BE} + \overline{n}_{FD}} < 0.01. \tag{13}$$

$$\begin{aligned} \overline{n}_{BE} - \overline{n}_{FD} &= \frac{1}{e^{(\epsilon-\mu)/kT}-1} - \frac{1}{e^{(\epsilon-\mu)/kT}+1} \\ &= \frac{2}{e^{2(\epsilon-\mu)/kT}-1}. \end{aligned} \tag{14}$$

And,

$$\begin{aligned} \overline{n}_{BE} + \overline{n}_{FD} &= \frac{1}{e^{(\epsilon-\mu)/kT}-1} + \frac{1}{e^{(\epsilon-\mu)/kT}+1} \\ &= \frac{2e^{(\epsilon-\mu)/kT}}{e^{2(\epsilon-\mu)/kT}-1}. \end{aligned} \tag{15}$$

So, the condition of closeness becomes,

$$\frac{2}{e^{(\epsilon-\mu)/kT}} < 0.01. \tag{16}$$

Hence,

$$e^{(\epsilon-\mu)/kT} > \frac{2}{0.01} = 200. \tag{17}$$

This gives,

$$(\epsilon-\mu)/kT > \ln 200. \tag{18}$$

So,

$$\epsilon - \mu > 1.38 \times 10^{-23} \times 300 \times \ln 200 = 2.29 \times 10^{-20}\ \text{J}. \tag{19}$$