Solutions
Chapter 4

Problem 4.1

Part a

In step$A$, the starting temperature is P_1V_1/Nk and the ending temperature is P_2V_1/Nk. So, the change in internal energy is (noting that $P_2 = 2P_1$),

$$(\Delta U)_A = (5Nk/2)\left(\frac{P_2V_1}{Nk} - \frac{P_1V_1}{Nk}\right) = (5/2)V_1(P_2 - P_1) = 5P_1V_1/2. \tag{1}$$

As this step is at constant volume, no work is done. Hence, the heat absorbed is the same as the change in internal energy.

$$Q_A = (\Delta U)_A = 5P_1V_1/2 \tag{2}$$

In step B, a similar computation for internal energy gives,

$$(\Delta U)_B = (5Nk/2)\left(\frac{P_2V_2}{Nk} - \frac{P_2V_1}{Nk}\right) = (5/2)P_2(V_2 - V_1) = 10P_1V_1. \tag{3}$$

The work done in this step is,

$$W_B = -\int_{V_1}^{V_2} P_2\, dV = -P_2\int_{V_1}^{V_2} dV = -P_2(V_2 - V_1) = -4P_1V_1. \tag{4}$$

Hence, the heat absorbed is,

$$Q_B = (\Delta U)_B - W_B = 10P_1V_1 - (-4P_1V_1) = 14P_1V_1. \tag{5}$$

In step C, a similar computation gives,

$$(\Delta U)_C = (5Nk/2)\left(\frac{P_1V_2}{Nk} - \frac{P_2V_2}{Nk}\right) = (5/2)V_2(P_1 - P_2) = -15P_1V_1/2. \tag{6}$$

The work done this step is again zero. So, the heat absorbed is the same as the change in internal energy.

$$Q_C = (\Delta U)_C = -15P_1V_1/2. \tag{7}$$

Finally, in step D,

$$(\Delta U)_D = (5Nk/2)\left(\frac{P_1V_1}{Nk} - \frac{P_1V_2}{Nk}\right) = (5/2)P_1(V_1 - V_2) = -5P_1V_1. \tag{8}$$

The work done is,

$$W_D = -\int_{V_2}^{V_1} P_1\, dV = -P_1\int_{V_2}^{V_1} dV = -P_1(V_1 - V_2) = 2P_1V_1. \tag{9}$$

So, the heat absorbed is,

$$Q_D = (\Delta U)_D - W_D = -7P_1V_1. \tag{10}$$

In the notation of the first law, W is the work done *on* the system. But here we need the work done *by* the system. Hence,

$$W = -(W_B + W_D) = 2P_1V_1. \tag{11}$$

The net heat input is,

$$Q_i = Q_A + Q_B = (14 + 5/2)P_1V_1. \tag{12}$$

So, the efficiency is,

$$e = \frac{W}{Q_i} = \frac{2}{14 + 5/2} = 0.12. \tag{13}$$

Note that, in this heat cycle as well as in real heat engines, there is no constant temperature heat reservoir. So, the heat input does not come from a reservoir at constant temperature T_h. However, for a practical definition of efficiency, we need the *total input heat* (input only – at any temperature) in the denominator. Here, I have called it Q_i. On the other hand, the numerator must always be *the net full cycle work output*.

Part b

The high temperature is,

$$T_h = \frac{P_2V_2}{Nk} = 6\frac{P_1V_1}{Nk}, \tag{14}$$

and the low temperature is,

$$T_c = \frac{P_1V_1}{Nk}, \tag{15}$$

Hence, the maximum efficiency is,

$$e = 1 - \frac{T_c}{T_h} = 1 - 1/6 = 0.83. \tag{16}$$

Problem 4.2

Part a

Maximum possible efficiency is,

$$e = 1 - \frac{T_c}{T_h} = 1 - \frac{273 + 20}{273 + 500} = 0.621. \tag{17}$$

Part b

At the higher temperature the efficiency is,

$$e_h = 1 - \frac{T_c}{T_h} = 1 - \frac{273 + 20}{273 + 600} = 0.664. \tag{18}$$

The ratio of efficiencies must be the ratio of powers generated. Hence,

$$\frac{P_h}{P} = \frac{e_h}{e}, \tag{19}$$

where P is the original power generated and P_h is the new higher power generated. Hence,

$$P_h = P\frac{e_h}{e}. \tag{20}$$

So, the extra power generated is,

$$P_e = P_h - P = P\left(\frac{e_h}{e} - 1\right) = 1.00\text{GW} \times 0.0692 = 6.92 \times 10^4 \text{ kW}. \tag{21}$$

The amount of extra energy produced in a year would be,

$$W_e = P_e \times 24 \times 365 = 6.92 \times 10^4 \times 24 \times 365 = 6.06 \times 10^8 \text{ kWh}. \tag{22}$$

At 5 cents per kWh, this gives,

$$6.06 \times 10^8 \times 0.05 = \$30,300,000. \tag{23}$$

Problem 4.5

Let us start at the point where the pressure is the lowest and the volume the highest and name them as the pair (P_1, V_1). Similarly, at the next three corners of the cycle, the pressure volume pairs can be named (P_2, V_2), (P_3, V_3) and (P_4, V_4). Hence, the work done on the engine in the first isothermal step at temperature T_c is,

$$W_{12} = -\int_{V_1}^{V_2} P\,dV = -NkT_c \int_{V_1}^{V_2} \frac{dV}{V} = -NkT_c \ln\frac{V_2}{V_1}. \tag{24}$$

As this is an isothermal process, there is no temperature change and hence, no internal energy change. So, the heat absorbed is,

$$Q_{12} = -W_{12} = NkT_c \ln\frac{V_2}{V_1}. \tag{25}$$

Similarly, the heat absorbed in the step from point 3 to 4 is,

$$Q_{34} = -W_{34} = NkT_h \ln\frac{V_4}{V_3}. \tag{26}$$

No heat is transferred in the adiabatic steps by definition. So, from the conservation of energy, the sum of Q_{12} and Q_{34} must be the work done by the engine (note that Q_{12} is negative). The heat input from the hot reservoir is Q_{34}. Hence, the efficiency is,

$$e = \frac{Q_{12} + Q_{34}}{Q_{34}}. \tag{27}$$

Now, the two adiabatic steps give the following.

$$P_2 V_2^\gamma = P_3 V_3^\gamma \quad \text{and} \quad P_1 V_1^\gamma = P_4 V_4^\gamma. \tag{28}$$

Using the ideal gas law these equations become as follows.

$$NkT_c V_2^{\gamma-1} = NkT_h V_3^{\gamma-1} \quad \text{and} \quad NkT_c V_1^{\gamma-1} = NkT_h V_4^{\gamma-1}. \tag{29}$$

This leads to the following.

$$\left(\frac{V_3}{V_2}\right)^{\gamma-1} = \frac{T_c}{T_h} \quad \text{and} \quad \left(\frac{V_4}{V_1}\right)^{\gamma-1} = \frac{T_c}{T_h}. \tag{30}$$

Hence,

$$\frac{V_3}{V_2} = \frac{V_4}{V_1} \quad \text{or} \quad \frac{V_2}{V_1} = \frac{V_3}{V_4}. \tag{31}$$

Then,

$$\ln\frac{V_2}{V_1} = \ln\frac{V_3}{V_4} = -\ln\frac{V_4}{V_3} \tag{32}$$

Using this in equations 25 and 26 and then inserting in equation 27 gives,

$$e = \frac{T_h - T_c}{T_h} = 1 - \frac{T_c}{T_h}. \tag{33}$$

Problem 4.7

If the air conditioner is inside the room, the heat Q_h is put back in the room while Q_c is being taken out from it. As $Q_h = Q_c + W$, this will add more heat to the room than it takes out. The extra heat is due to W the work done by the power source. This is clearly counterproductive!

Problem 4.8

An open refrigerator is exactly like an air conditioner placed inside a room. So, the result is the same as in problem 4.7.

Problem 4.10

If the leaky refrigerator is in a steady state, i.e. the amount of heat pumped out of the refrigerator (Q_c) and the amount of heat deposited to the outside (Q_h) do not change with time, then the net heat that leaks back into the refrigerator is $Q_h - Q_c$. Due to energy conservation,

$$W = Q_h - Q_c. \tag{34}$$

As W is the external work done on the system, it is also the leakage energy. So the rate at which power is drawn from the wall outlet must also be the leakage rate – 300 watts.

Problem 4.18

Consider figure 4.5 of the textbook for this problem. The work done on the system in going from point 1 to point 2 is,

$$W_{12} = -\int_{V_1}^{V_2} P \, dV. \tag{35}$$

As this is an adiabatic process, there is no heat input and,

$$P_1 V_1^\gamma = P_2 V_2^\gamma = K, \tag{36}$$

where K is a constant. So, the above integral can be written as,

$$W_{12} = -K \int_{V_1}^{V_2} V^{-\gamma} \, dV = -\frac{K}{1-\gamma}(V_2^{1-\gamma} - V_1^{1-\gamma}). \tag{37}$$

Replacing K by $P_2 V_2^\gamma$ (using equation 36) gives,

$$W_{12} = -\frac{P_2 V_2^\gamma}{1-\gamma}(V_2^{1-\gamma} - V_1^{1-\gamma}) = -\frac{P_2 V_2}{1-\gamma}\left(1 - \left(\frac{V_1}{V_2}\right)^{1-\gamma}\right). \tag{38}$$

Similarly, the work done in the adiabatic step from point 3 to point 4 can be written as follows.

$$W_{34} = -J \int_{V_3}^{V_4} V^{-\gamma} \, dV = -\frac{J}{1-\gamma}(V_4^{1-\gamma} - V_3^{1-\gamma}), \tag{39}$$

where J comes from the following adiabatic equation.

$$P_3 V_3^\gamma = P_4 V_4^\gamma = J,. \tag{40}$$

So, using $J = P_3 V_3^\gamma$ and the given information that $V_2 = V_3$ and $V_4 = V_1$, we get,

$$W_{34} = -\frac{P_3 V_2^\gamma}{1-\gamma}(V_1^{1-\gamma} - V_2^{1-\gamma}) = -\frac{P_3 V_2}{1-\gamma}\left(\left(\frac{V_1}{V_2}\right)^{1-\gamma} - 1\right) \tag{41}$$

As W_{12} and W_{34} are the works done *on* the system, the total work done *by* the system is,

$$W = -(W_{12} + W_{34}) = \frac{(P_3 - P_2)V_2}{1-\gamma}\left(\left(\frac{V_1}{V_2}\right)^{1-\gamma} - 1\right). \tag{42}$$

As the volume is the same at points 2 and 3, the ideal gas equation gives,

$$P_2V_2 = NkT_2 \quad \text{and} \quad P_3V_2 = NkT_3. \tag{43}$$

Hence,

$$P_3 - P_2 = \frac{Nk(T_3 - T_2)}{V_2}. \tag{44}$$

The heat input in the step from point 2 to 3 is,

$$Q_i = C_V(T_3 - T_2). \tag{45}$$

Using this in equation 44 we get,

$$P_3 - P_2 = \frac{NkQ_i}{C_V V_2}. \tag{46}$$

Note that, for an ideal gas $C_V = Nk/(\gamma - 1)$. Hence,

$$P_3 - P_2 = \frac{(\gamma - 1)Q_i}{V_2}. \tag{47}$$

Inserting this in equation 42 gives,

$$W = Q_i\left(1 - \left(\frac{V_1}{V_2}\right)^{1-\gamma}\right) = Q_i\left(1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1}\right). \tag{48}$$

Hence, the efficiency is,

$$e = \frac{W}{Q_i} = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1}. \tag{49}$$