Solutions
Chapter 2

Problem 2.3

Part a

The number is,

$$2^{50} \approx 1.1259 \times 10^{15}. \tag{1}$$

Part b

$$\Omega(N, n) = \binom{N}{n} = \frac{N!}{(N-n)!n!} = \frac{50!}{25!25!} \approx 1.2641 \times 10^{14} \tag{2}$$

Part c

Probability of 25 heads and 25 tails is,

$$P(N, n) = P(50, 25) = \Omega(50, 25)/2^{50} = 0.1123. \tag{3}$$

Part d

$$\Omega(50, 30) = \frac{50!}{30!20!} = 4.7129 \times 10^{13} \tag{4}$$

Hence,

$$P(50, 30) = \Omega(50, 30)/2^{50} = 0.04186. \tag{5}$$

Part e

$$\Omega(50, 40) = \frac{50!}{40!10!} = 1.0272 \times 10^{10} \tag{6}$$

Hence,

$$P(50, 40) = \Omega(50, 40)/2^{50} = 9.1236 \times 10^{-6}. \tag{7}$$

Part f

$$\Omega(50, 50) = \frac{50!}{50!0!} = 1. \tag{8}$$

Hence,

$$P(50, 50) = \Omega(50, 50)/2^{50} = 8.8818 \times 10^{-16}. \tag{9}$$

Problem 2.6

$$\Omega(N, q) = \binom{q + N - 1}{q} = \frac{(30 + 30 - 1)!}{30!(30 - 1)!} = 5.9132 \times 10^{16}. \tag{10}$$

Problem 2.8

Part a

As q_A can take values from 0 to 20, there must be 21 macrostates.

Part b

The number of microstates is,

$$\Omega(10 + 10, 20) = \frac{(20 + 20 - 1)!}{20!(20 - 1)!} = 6.8923 \times 10^{10}. \tag{11}$$

Part c

The number of microstates with all 20 units in solid A is,

$$\Omega(N_A, 20) = \Omega(10, 20) = \frac{(10 + 20 - 1)!}{20!(10 - 1)!} = 1.0015 \times 10^{7}. \tag{12}$$

Using the value for total number of microstates from part b, the probability is,

$$\frac{1.0015 \times 10^{7}}{6.8923 \times 10^{10}} = 1.4531 \times 10^{-4}. \tag{13}$$

Part d

The number of microstates with 10 units in A 10 units in B is,

$$\Omega = \Omega_A \Omega_B = \frac{(10+10-1)!}{10!(10-1)!} \frac{(10+10-1)!}{10!(10-1)!} = 8.5337 \times 10^9. \tag{14}$$

Using the result of part b, the probability is,

$$\frac{8.5337 \times 10^9}{6.8923 \times 10^{10}} = 0.1238. \tag{15}$$

Part e

It shows irreversible behavior when the energy is not equally distributed between the two solids.

Problem 2.12

Part b

Using the definition,

$$e^{\ln ab} = ab = e^{\ln a} e^{\ln b} = e^{\ln a + \ln b}. \tag{16}$$

Matching the power of e,

$$\ln ab = \ln a + \ln b. \tag{17}$$

Also using the definition,

$$e^{\ln a^b} = a^b = (e^{\ln a})^b = e^{b \ln a} \tag{18}$$

Once again, matching the power of e,

$$\ln a^b = b \ln a. \tag{19}$$

Part c

Let $y = \ln x$. Then, as $x = e^y$ using the chain rule,

$$1 = \frac{dx}{dx} = \frac{d}{dx} e^y = \frac{d}{dy} e^y \frac{dy}{dx} = e^y \frac{dy}{dx} = x \frac{dy}{dx}. \tag{20}$$

Hence,

$$\frac{dy}{dx} = \frac{1}{x}. \tag{21}$$

Part d

For the Taylor series expansion of the function,

$$f(x) = \ln(1+x), \tag{22}$$

about $x = 0$, the derivatives are,

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2, \quad f''''(0) = -6, \cdots \tag{23}$$

Hence,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{24}$$

For $|x| \ll 1$, the higher powers are much smaller and as an approximation one may use,

$$\ln(1+x) \approx x. \tag{25}$$

Problem 2.13

Part a

$$e^{a \ln b} = e^{(\ln b)a} = (e^{\ln b})^a = b^a. \tag{26}$$

Part b

If $b \ll a$ then $b/a \ll 1$. Hence, using the result of the last problem,

$$\ln(a+b) = \ln(a(1+b/a)) = \ln a + \ln(1+b/a) \approx \ln a + b/a. \tag{27}$$

Problem 2.17

The derivation is the same until equation 2.18 of the textbook. The approximation changes when the premise of $q \ll N$ is introduced. Then,

$$\ln(q+N) = \ln(N(1+q/N)) = \ln N + \ln(1+q/N) \approx \ln N + q/N. \tag{28}$$

So,

$$\begin{aligned}
\ln \Omega &= (q+N)\ln(q+N) - q\ln q - N\ln N \\
&\approx (q+N)(\ln N + q/N) - q\ln q - N\ln N \\
&= q\ln N + q - q\ln q + q^2/N \\
&\approx q\ln N + q - q\ln q \\
&= q\ln(N/q) + q.
\end{aligned} \tag{29}$$

Hence,

$$\Omega = e^{q\ln(N/q)}e^q = (N/q)^q e^q = \left(\frac{eN}{q}\right)^q. \tag{30}$$

Problem 2.23

Part a

Let $N = 10^{23}$. Then, the number of accessible microstates for half the dipoles to point up and the rest point down is,

$$\Omega = \frac{N!}{(N/2)!(N/2)!}. \tag{31}$$

Then,

$$\begin{aligned} \ln\Omega &\approx N\ln N - N - (N/2)\ln(N/2) + N/2 - (N/2)\ln(N/2) + N/2 \\ &= N\ln N - N\ln(N/2) \\ &= N\ln N - N(\ln N - \ln 2) \\ &= N\ln 2. \end{aligned} \tag{32}$$

Hence,

$$\Omega \approx e^{N\ln 2} = 2^N = 2^{10^{23}} = 10^{(\log 2)\times 10^{23}} = 10^{0.3\times 10^{23}}. \tag{33}$$

Part b

The number of times it changes per year is,

$$10^9 \times 60 \times 60 \times 24 \times 365 = 3.1536 \times 10^{16}. \tag{34}$$

So, in ten billion years it will change

$$3.1536 \times 10^{16} \times 10^{10} = 3.1536 \times 10^{26} \text{times}. \tag{35}$$

This is still negligibly smaller than Ω from part a.

Problem 24

Part a

As shown in part a of problem 23,

$$\Omega_{\text{max}} \approx 2^N. \tag{36}$$

Part b

The multiplicity is,

$$\Omega = \frac{N!}{N_\uparrow! N_\downarrow!}, \tag{37}$$

where,

$$N_\uparrow = N/2 + x, \quad \text{and} \quad N_\downarrow = N/2 - x. \tag{38}$$

Using Sterling's approximation,

$$\ln \Omega \approx N \ln N - N - N_\uparrow \ln N_\uparrow + N_\uparrow - N_\downarrow \ln N_\downarrow + N_\downarrow. \tag{39}$$

As $x \ll N/2$, one may approximate

$$\ln N_\uparrow = \ln(N/2 + x) = \ln((N/2)(1 + 2x/N)) \approx \ln(N/2) + 2x/N, \tag{40}$$

and,

$$N_\uparrow \ln N_\uparrow \approx (N/2 + x)(\ln(N/2) + 2x/N) = (N/2)\ln(N/2) + x + \ln(N/2)x + 2x^2/N. \tag{41}$$

Similarly,

$$N_\downarrow \ln N_\downarrow \approx (N/2)\ln(N/2) - x - \ln(N/2)x + 2x^2/N. \tag{42}$$

Inserting equations 41 and 42 in equation 39 and noting that $N_\uparrow + N_\downarrow = N$ gives,

$$\ln \Omega \approx N \ln N - N \ln(N/2) - 4x^2/N = N \ln 2 - 4x^2/N. \tag{43}$$

Hence,

$$\Omega \approx e^{N(\ln 2)} e^{-4x^2/N} = 2^N e^{-4x^2/N}. \tag{44}$$

Part c

As Ω falls off from its maximum by a factor of $1/e$ at $x = \sqrt{N}/2$, the width is twice of that:

$$\text{width} = \sqrt{N}. \tag{45}$$

Problem 29

For the most likely macrostate $(q_A = 60, q_B = 40)$

$$\Omega = 6.9 \times 10^{114}, \quad \text{hence,} \quad S = \ln \Omega = 264. \tag{46}$$

For the least likely macrostate $(q_A = 0, q_B = 100)$

$$\Omega = 2.8 \times 10^{81}, \quad \text{hence,} \quad S = \ln \Omega = 188. \tag{47}$$

For all microstates

$$\Omega = 9.3 \times 10^{115}, \quad \text{hence,} \quad S = \ln \Omega = 267. \tag{48}$$

Problem 34

For an ideal gas U does not change in an isothermal process. Hence, $\Delta U = 0$. Then, using the first law,

$$Q = -W = \int_{V_i}^{V_f} P\, dV = NkT \int_{V_i}^{V_f} \frac{dV}{V} = NkT \ln \frac{V_f}{V_i} = T\Delta S. \tag{49}$$

In the last step, equation 2.51 of the textbook is used. Hence,

$$\Delta S = \frac{Q}{T}. \tag{50}$$