Solutions
Chapter 4

Problem 1

Part a

The velocity of the s^{th} atom is du_s/dt. Hence, its kinetic energy is,

$$K_s = \frac{1}{2} M \left(\frac{du_s}{dt} \right)^2 .$$

Then the total kinetic energy is,

$$K = \sum_s K_s = \frac{1}{2} M \sum_s \left(\frac{du_s}{dt} \right)^2 .$$

The potential energy is like that of springs between atoms. The spring extension of the spring between the s^{th} and the $(s+1)^{\text{th}}$ atom can be represented by $x_s = u_s - u_{s+1}$. The spring constant is C. Then the potential energy of this spring is,

$$U_s = \frac{1}{2} C x_s^2 = \frac{1}{2} C (u_s - u_{s+1})^2 .$$

And the total potential energy is,

$$U = \sum_s U_s = \frac{1}{2} C \sum_s (u_s - u_{s+1})^2 .$$

This gives the total energy to be,

$$E = K + U = \frac{1}{2} M \sum_s \left(\frac{du_s}{dt} \right)^2 + \frac{1}{2} C \sum_s (u_s - u_{s+1})^2 .$$

Part b

$$\left(\frac{du_s}{dt} \right)^2 = (-\omega u \sin(\omega t - sKa))^2 = \omega^2 u^2 \sin^2(\omega t - sKa).$$

The time average of this is,

$$\left(\frac{du_s}{dt} \right)^2_{\text{av}} = \omega^2 u^2 \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \sin^2(\omega t - sKa)\, dt = \omega^2 u^2 / 2.$$

So, the time average kinetic energy per atom is,

$$K_{s\text{av}} = \frac{1}{2}M\left(\frac{du_s}{dt}\right)^2_{\text{av}} = \frac{1}{4}M\omega^2u^2.$$

$$(u_s - u_{s+1})^2 = u^2(\cos(\omega t - sKa) - \cos(\omega t - sKa - Ka))^2 = u^2(-2\sin(\omega t - sKa - Ka/2)\sin(Ka/2))^2$$
$$= 4u^2\sin^2(\omega t - (s+1)sKa)\sin^2(Ka/2).$$

The time average of this is,

$$(u_s - u_{s+1})^2_{\text{av}} = 4u^2\sin^2(Ka/2)\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sin^2(\omega t - (s+1)sKa)\,dt = 2u^2\sin^2(Ka/2).$$

So, the time average potential energy per atom is,

$$U_{s\text{av}} = \frac{1}{2}C(u_s - u_{s+1})^2_{\text{av}} = Cu^2\sin^2(Ka/2) = \frac{Cu^2}{2}(1-\cos(Ka)).$$

Then the total time average energy per atom is,

$$K_{s\text{av}} + U_{s\text{av}} = \frac{1}{4}M\omega^2u^2 + \frac{Cu^2}{2}(1-\cos(Ka)) = \frac{1}{2}M\omega^2u^2,$$

as the dispersion relation gives,

$$\omega^2 = \frac{2C}{M}(1-\cos(Ka)).$$

Problem 3

For $K = \pi/a$,

$$\exp(iKa) = \exp(-iKa) = -1.$$

Hence equation (20) of the book gives,

$$\omega^2 M_1 u = 2Cu,$$

$$\omega^2 M_2 v = 2Cv$$

For $M_1 > M_2$ the acoustical branch has $\omega^2 = 2C/M_1$. Hence, the solution of the above equation gives $u \neq 0$ and $v = 0$. Thus $u/v = \infty$. For $M_1 > M_2$ the optical branch has $\omega^2 = 2C/M_2$. Hence, the solution of the above equation gives $u = 0$ and $v \neq 0$. Thus $u/v = 0$. So, for the first case, atoms of mass M_1 move but those of mass M_2 don't. For the second case atoms of mass M_2 move but those of mass M_1 don't.