Solutions
Chapter 4

Problem 9

Part a

$$n_1(x) = -(-x)\frac{1}{x}\frac{d}{dx}\frac{\cos x}{x} = -\frac{\cos x}{x^2} - \frac{\sin x}{x}.$$

$$\begin{aligned}
n_2(x) = -(-x)^2\left(\frac{1}{x}\frac{d}{dx}\right)^2\frac{\cos x}{x} &= -x^2\frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\frac{\cos x}{x}\right) = -x\frac{d}{dx}\left(\frac{1}{x}\left(-\frac{\cos x}{x^2} - \frac{\sin x}{x}\right)\right) \\
&= x\frac{d}{dx}\left(\frac{\cos x}{x^3} + \frac{\sin x}{x^2}\right) = x\left(-3\frac{\cos x}{x^4} - \frac{\sin x}{x^3} - 2\frac{\sin x}{x^3} + \frac{\cos x}{x^2}\right) \\
&= -3\frac{\cos x}{x^3} - 3\frac{\sin x}{x^2} + \frac{\cos x}{x}.
\end{aligned}$$

Part b

For $x \ll 1$,

$$n_1(x) \approx -\frac{1}{x^2}(1 - x^2/2) - \frac{x}{x} = -\frac{1}{x^2} - 1/2.$$

This, of course, blows up at the origin.

For $x \ll 1$,

$$\begin{aligned}
n_2(x) &\approx -\frac{3}{x^3}(1 - x^2/2 + x^4/24) - \frac{3}{x^2}(x - x^3/6) + \frac{1}{x}(1 - x^2/2) \\
&\approx -\frac{3}{x^3} + \frac{3}{2x} - \frac{3}{x} + \frac{1}{x} = -\frac{3}{x^3} - \frac{1}{2x}.
\end{aligned}$$

This too blows up at the origin.

Problem 15

Part a

The ground state wavefunction is

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}.$$

Hence,

$$\langle r \rangle = \frac{1}{\pi a^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} e^{-r/a} r e^{-r/a} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{4\pi}{\pi a^3} \int_0^{\infty} r^3 e^{-2r/a} dr$$

$$= \frac{4\pi}{\pi a^3} \frac{3a^4}{8} = 3a/2.$$

$$\langle r^2 \rangle = \frac{1}{\pi a^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} e^{-r/a} r^2 e^{-r/a} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{4\pi}{\pi a^3} \int_0^{\infty} r^4 e^{-2r/a} dr$$

$$= \frac{4\pi}{\pi a^3} \frac{3a^5}{4} = 3a^2.$$

Part b

$$\langle x \rangle = \langle r \sin\theta \cos\phi \rangle = \frac{1}{\pi a^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} e^{-r/a} r \sin\theta \cos\phi \, e^{-r/a} r^2 \sin\theta \, dr \, d\theta \, d\phi = 0,$$

as

$$\int_0^{2\pi} \cos\phi \, d\phi = 0.$$

As the ground state is spherically symmetric,

$$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle,$$

and hence,

$$\langle r^2 \rangle = \langle (x^2 + y^2 + z^2) \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = 3 \langle x^2 \rangle.$$

So,

$$\langle x^2 \rangle = \langle r^2 \rangle / 3 = a^2,$$

using the computation of part a.

Part c

$$\psi_{211} = R_{21} Y_1^1 = -\frac{a^{-3/2}}{2\sqrt{6}} \left(\frac{r}{a} \right) \exp(-r/2a) \left(\frac{3}{8\pi} \right)^{1/2} \sin\theta \, e^{i\phi}$$

$$|\psi_{211}|^2 = \frac{1}{64\pi a^5} r^2 \exp(-r/a) \sin^2\theta$$

$$\langle x^2 \rangle = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} x^2 |\psi_{211}|^2 r^2 \sin\theta \, dr \, d\theta \, d\phi = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} r^2 \sin^2\theta \cos^2\phi \, |\psi_{211}|^2 r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{1}{64\pi a^5} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} r^6 \sin^5\theta \cos^2\phi \, \exp(-r/a)\, dr\, d\theta\, d\phi$$

$$= \frac{1}{64\pi a^5} \int_0^{2\pi} \cos^2\phi \, d\phi \int_0^{\pi} \sin^5\theta \, d\theta \int_0^{\infty} r^6 \, \exp(-r/a)\, dr$$

$$= \frac{1}{64\pi a^5} (\pi)(16/15)(6! a^7) = 12a^2.$$

Problem 19

Substituting Ze^2 for e^2,

$$E_n(Z) = -\left[\frac{m_e}{2\hbar^2}\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1(Z)}{n^2},$$

where

$$E_1(Z) = -\left[\frac{m_e}{2\hbar^2}\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2\right].$$

$$a(Z) = \frac{4\pi\epsilon_0\hbar^2}{m_e Z e^2}.$$

$$\mathcal{R}(Z) = \frac{m_e}{4\pi c\hbar^3}\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2.$$

Problem 22

Part a

$$[L_z, x] = [(xp_y - yp_x), x] = [xp_y, x] - [yp_x, x] = [x, x]\, p_y + x\,[p_y, x] - y\,[p_x, x] - [y, x]\, p_x$$

$$= -y\,[p_x, x] = i\hbar y.$$

$$[L_z, y] = [(xp_y - yp_x), y] = [xp_y, y] - [yp_x, y] = [x, y]\, p_y + x\,[p_y, y] - y\,[p_x, y] - [y, y]\, p_x$$

$$= x\,[p_y, y] = -i\hbar x.$$

$$[L_z, z] = [(xp_y - yp_x), z] = [xp_y, z] - [yp_x, z] = [x, z]\, p_y + x\,[p_y, z] - y\,[p_x, z] - [y, z]\, p_x = 0.$$

$$[L_z, p_x] = [(xp_y - yp_x), p_x] = [xp_y, p_x] - [yp_x, p_x] = [x, p_x]\, p_y + x\, [p_y, p_x] - y\, [p_x, p_x] - [y, p_x]\, p_x$$
$$= [x, p_x]\, p_y = i\hbar p_y.$$

$$[L_z, p_y] = [(xp_y - yp_x), p_y] = [xp_y, p_y] - [yp_x, p_y] = [x, p_y]\, p_y + x\, [p_y, p_y] - y\, [p_x, p_y] - [y, p_y]\, p_x$$
$$= -\, [y, p_y]\, p_x = -i\hbar p_x.$$

$$[L_z, p_z] = [(xp_y - yp_x), p_z] = [xp_y, p_z] - [yp_x, p_z] = [x, p_z]\, p_y + x\, [p_y, p_z] - y\, [p_x, p_z] - [y, p_z]\, p_x = 0.$$

Part b

$$[L_z, L_x] = [L_z, (yp_z - zp_y)] = [L_z, yp_z] - [L_z, zp_y] = y\, [L_z, p_z] + [L_z, y]\, p_z - z\, [L_z, p_y] - [L_z, z]\, p_y$$
$$= -i\hbar xp_z + i\hbar zp_x = i\hbar L_y.$$

Part c

$$[L_z, x^2] = x[L_z, x] + [L_z, x]x = 2i\hbar xy.$$
$$[L_z, y^2] = y[L_z, y] + [L_z, y]y = -2i\hbar xy.$$
$$[L_z, z^2] = z[L_z, z] + [L_z, z]z = 0.$$

Hence,

$$[L_z, r^2] = [L_z, x^2] + [L_z, y^2] + [L_z, z^2] = 0.$$

$$[L_z, p_x^2] = p_x[L_z, p_x] + [L_z, p_x]p_x = 2i\hbar p_x p_y.$$
$$[L_z, p_y^2] = p_y[L_z, p_y] + [L_z, p_y]p_y = -2i\hbar p_x p_y.$$
$$[L_z, p_z^2] = p_z[L_z, p_z] + [L_z, p_z]p_z = 0.$$

Hence,

$$[L_z, p^2] = [L_z, p_x^2] + [L_z, p_y^2] + [L_z, p_z^2] = 0.$$

Part d

As r^2 and p^2 are spherically symmetric, their commuting with L_z means they commute with L_x and L_y as well. Hence,

$$[L_x, r^2] = [L_y, r^2] = [L_z, r^2] = [L_x, p^2] = [L_y, p^2] = [L_z, p^2] = 0.$$

If V is a function of r alone, the above result means

$$[L_x, V] = [L_y, V] = [L_z, V] = 0.$$

Also, as $(p^2/2m)$ is a function of p^2 alone, the above result means

$$[L_x, (p^2/2m)] = [L_y, (p^2/2m)] = [L_z, (p^2/2m)] = 0.$$

Then, it follows that

$$[L_x, H] = [L_y, H] = [L_z, H] = 0.$$

Hence,

$$[L^2, H] = [L_x^2 + L_y^2 + L_z^2, H] = [L_x^2, H] + [L_y^2, H] + [L_z^2, H] = 0.$$

Problem 25

Part a

Note that Y_l^l is Y_l^m with $m = l$ (its maximum value) So, an operation by L_+ cannot raise the m value any further. hence,

$$L_+ Y_l^l = 0$$

Part b

The above equation gives

$$\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) Y_l^l = 0. \tag{1}$$

However, as $L_z Y_l^l = \hbar l Y_l^l$ and $L_z = -i\hbar\partial/\partial\phi$,

$$\frac{\partial Y_l^l}{\partial\phi} = il Y_l^l.$$

This gives,

$$Y_l^l = \Theta(\theta)e^{il\phi}, \tag{2}$$

where $\Theta(\theta)$ is, so far, an undetermined function of θ. Substituting equation 2 in equation 1, we get

$$\frac{d\Theta}{d\theta} - l(\cot\theta)\Theta = 0.$$

or,

$$\frac{d\Theta}{\Theta} = l \cot\theta \ \ d\theta$$

Integrating gives,

$$\ln\Theta = \ln(\sin\theta)^l + \ln A.$$

Hence,

$$\Theta = A(\sin\theta)^l$$

So, from equation 2,

$$Y_l^l = A(\sin\theta)^l e^{il\phi}$$

Part c

$$\begin{aligned}
1 &= \int_0^{2\pi}\int_0^{\pi} (Y_l^l)^* Y_l^l \sin\theta \, d\theta \, d\phi = |A|^2 \int_0^{2\pi}\int_0^{\pi} \sin^{2l}\theta \sin\theta \, d\theta \, d\phi \\
&= 2\pi|A|^2 \int_0^{\pi} \sin^{2l+1}\theta \, d\theta = 2\pi|A|^2 \frac{2l}{2l+1}\int_0^{\pi} \sin^{2l-1}\theta \, d\theta \\
&= 2\pi|A|^2 \frac{2l(2l-2)}{(2l+1)(2l-1)}\int_0^{\pi} \sin^{2l-3}\theta \, d\theta \\
&= \cdots = 2\pi|A|^2 \frac{2l(2l-2)\cdots 2}{(2l+1)(2l-1)\cdots 3}\int_0^{\pi} \sin\theta \, d\theta = 2\pi|A|^2 \frac{2l(2l-2)\cdots 2}{(2l+1)(2l-1)\cdots 3}(2) \\
&= 4\pi|A|^2 \frac{2l(2l-2)\cdots 2}{(2l+1)(2l-1)\cdots 3} = 4\pi|A|^2 \frac{(2l(2l-2)\cdots 2)^2}{(2l+1)!} = 4\pi|A|^2 \frac{(2^l l!)^2}{(2l+1)!}.
\end{aligned}$$

Hence,

$$A = \frac{1}{2^{l+1} l!}\sqrt{\frac{(2l+1)!}{\pi}}.$$

Problem 27

Part a

The distance of mass m_1 from the center of mass is

$$r_1 = \frac{m_2 a}{m_1 + m_2},$$

and the distance of mass m_2 from the center of mass is

$$r_2 = \frac{m_1 a}{m_1 + m_2}.$$

As the angular speed ω is the same for both masses, their total angular momentum magnitude is

$$L = m_1 r_1^2 \omega + m_2 r_2^2 \omega = \frac{m_1 m_2 a^2 \omega}{m_1 + m_2} = I\omega.$$

The total energy of the two masses is

$$E = \frac{1}{2} m_1 (r_1 \omega)^2 + \frac{1}{2} m_2 (r_2 \omega)^2 = \frac{1}{2} L\omega = \frac{1}{2I} L^2,$$

using the equation above.

So, the quantum hamiltonian is

$$H = \frac{L^2}{2I}.$$

As the eigenvalues of L^2 are $\hbar^2 n(n+1)$ with $n = 0, 1, 2, \ldots$, the energy eigenvalues are

$$E_n = \frac{\hbar^2}{2I} n(n+1), \quad n = 0, 1, 2, \ldots.$$

Part b

The eigenfunctions must be the same as those of L^2 – the spherical harmonics $Y_n^m(\theta, \phi)$. The degeneracy must also be the same – $(2n + 1)$.

Part c

Consider two adjacent levels with $n = j$ and $n = j - 1$. Their energy difference is

$$\Delta E = E_j - E_{j-1} = \frac{\hbar^2}{2I}(j(j+1) - (j-1)(j-1+1)) = \frac{\hbar^2}{2I}(2j) = \frac{\hbar^2 j}{I}.$$

The photon frequency is related to the energy difference as

$$\nu_j = \Delta E / h = \frac{\hbar j}{2\pi I}.$$

Problem 29

Part a

$$[S_x, S_y] = \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]$$

$$= \frac{\hbar^2}{4} \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right] = \frac{\hbar^2}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\hbar S_z.$$

Similarly, the others can also be shown.

Part b

$$\sigma_x\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1,$$

where "1" represents the identity matrix. Similarly,

$$\sigma_y\sigma_y = \sigma_z\sigma_z = 1.$$

$$\sigma_x\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_z.$$

Similarly,

$$\sigma_y\sigma_z = i\sigma_x, \quad \text{and} \quad \sigma_z\sigma_x = i\sigma_y.$$

Also,

$$\sigma_y\sigma_x = -i\sigma_z, \quad \sigma_z\sigma_y = -i\sigma_x, \quad \text{and} \quad \sigma_x\sigma_z = -i\sigma_y.$$

Then, from the defined properties of the Levi-Civita symbol,

$$\sigma_j\sigma_k = \delta_{jk} + i\sum_l \epsilon_{jkl}\sigma_l.$$

Problem 30

Part a

$$1 = |A|^2 \begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = 25|A|^2.$$

Hence,

$$A = 1/5.$$

Part b

$$\langle S_x \rangle = |A|^2\frac{\hbar}{2}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = |A|^2\frac{\hbar}{2}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 3i \end{pmatrix} = 0.$$

$$\langle S_y \rangle = |A|^2\frac{\hbar}{2}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = |A|^2\frac{\hbar}{2}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} -4i \\ -3 \end{pmatrix} = |A|^2\frac{\hbar}{2}(-24) = -\frac{12\hbar}{25}.$$

$$\langle S_z \rangle = |A|^2\frac{\hbar}{2}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = |A|^2\frac{\hbar}{2}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 3i \\ -4 \end{pmatrix} = |A|^2\frac{\hbar}{2}(-7) = -\frac{7\hbar}{50}.$$

Part c

As

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1,$$

we see that,

$$S_x^2 = S_y^2 = S_z^2 = \frac{\hbar^2}{4}1,$$

"1" being the identity matrix. Hence,

$$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4}$$

This gives,

$$\sigma_{S_x}^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4}.$$

$$\sigma_{S_y}^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} - \left(\frac{12\hbar}{25}\right)^2 = \frac{49}{2500}\hbar^2.$$

$$\sigma_{S_z}^2 = \langle S_z^2 \rangle - \langle S_z \rangle^2 = \frac{\hbar^2}{4} - \left(\frac{7}{50}\right)^2 \hbar^2 = \frac{576}{2500}\hbar^2.$$

Hence,

$$\sigma_{S_x} = \frac{\hbar}{2}, \quad \sigma_{S_y} = \frac{7\hbar}{50}, \quad \sigma_{S_z} = \frac{12\hbar}{25}.$$

Part d

$$\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\frac{7\hbar}{50} = \frac{\hbar}{2}|\langle S_z \rangle|.$$

$$\sigma_{S_y}\sigma_{S_z} = \frac{7\hbar}{50}\frac{12\hbar}{25} \geq \frac{\hbar}{2}|\langle S_x \rangle| = 0.$$

$$\sigma_{S_z}\sigma_{S_x} = \frac{12\hbar}{25}\frac{\hbar}{2} = \frac{\hbar}{2}|\langle S_y \rangle|.$$

Problem 34

For spin 1, the possible values for m are: $-1, 0, 1$. Hence, the three basis states are:

$$\chi_+ = |1, 1\rangle, \quad \chi_0 = |1, 0\rangle \quad \chi_- = |1, -1\rangle.$$

With these as bases, their spinor notation is given as follows.

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvalues of S^2 are expected to be the same for the above three eigenstates: $2\hbar^2$. Hence,

$$S^2\chi_+ = 2\hbar^2\chi_+, \quad S^2\chi_0 = 2\hbar^2\chi_0, \quad S^2\chi_- = 2\hbar^2\chi_-.$$

Let S^2 be written as the following matrix.

$$S^2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then the first eigenvalue equation above gives,

$$S^2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This gives,

$$a_{11} = 2\hbar^2, \quad a_{21} = 0, \quad a_{31} = 0.$$

Similarly, the other two eigenvalue equations give,

$$a_{12} = 0, \quad a_{22} = 2\hbar^2, \quad a_{32} = 0, \quad a_{13} = 0, \quad a_{23} = 0, \quad a_{33} = 2\hbar^2.$$

Hence,

$$S^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Next, we look at the following three eigenvalue equations.

$$S_z\chi_+ = \hbar\chi_+, \quad S_z\chi_0 = 0, \quad S_z\chi_- = -\hbar\chi_-.$$

A derivation similar to that of S^2, gives,

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For the operators S_+ and S_-, equation 4.136 of the textbook gives

$$S_+\chi_+ = 0, \quad S_+\chi_0 = \sqrt{2}\hbar\chi_+, \quad S_+\chi_- = \sqrt{2}\hbar\chi_0.$$

$$S_-\chi_+ = \sqrt{2}\hbar\chi_0, \quad S_-\chi_0 = \sqrt{2}\hbar\chi_-, \quad S_-\chi_- = 0.$$

A similar evaluation method as for S^2 and S_z gives

$$S_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence,

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

Problem 35

Part a

The eigenstate of S_x for the eigenvalue $\hbar/2$ is

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, the probability of measuring $\hbar/2$ for the x spin is

$$|\chi(t)^\dagger \chi_+^{(x)}|^2 = \frac{1}{2}|\sin(\alpha/2)e^{i\gamma B_0 t/2} + \cos(\alpha/2)e^{-i\gamma B_0 t/2}|^2 = \frac{1}{2}(1 + \sin(\alpha)\cos(\gamma B_0 t)).$$

The eigenstate of S_y for the eigenvalue $\hbar/2$ is

$$\chi_+^{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Hence, the probability of measuring $\hbar/2$ for the y spin is

$$|\chi(t)^\dagger \chi_+^{(y)}|^2 = \frac{1}{2}|i\sin(\alpha/2)e^{i\gamma B_0 t/2} + \cos(\alpha/2)e^{-i\gamma B_0 t/2}|^2 = \frac{1}{2}(1 - \sin(\alpha)\sin(\gamma B_0 t)).$$

The eigenstate of S_z for the eigenvalue $\hbar/2$ is

$$\chi_+^{(z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence, the probability of measuring $\hbar/2$ for the z spin is

$$|\chi(t)^\dagger \chi_+^{(z)}|^2 = |\cos(\alpha/2)e^{-i\gamma B_0 t/2}|^2 = \cos^2(\alpha/2).$$