# Solutions

# Chapter 3

## Problem 2

### Part a

$$\int_0^1 |f(x)|^2 dx = \int_0^1 x^{2\nu} dx = \left.\frac{x^{2\nu+1}}{2\nu+1}\right|_0^1 = \frac{1}{2\nu+1}(1-0^{2\nu+1}).$$

This is finite for $(2\nu+1) > 0$, that is $\nu > -1/2$. For $\nu < -1/2$, the integral blows up. For $\nu = -1/2$,

$$\int_0^1 |f(x)|^2 dx = \int_0^1 \frac{1}{x} dx = \ln x|_0^1 = \infty.$$

Hence, the given function is in Hilbert space if $\nu > -1/2$.

### Part b

From the above result, $f(x)$ is in Hilbert space for $\nu = 1/2$. The function $xf(x) = x^{3/2}$, is also in Hilbert space as seen from above. However, $df(x)/dx = x^{-1/2}/2$ is not in Hilbert space as seen above.

## Problem 4

### Part a

Let $\hat{Q}$ and $\hat{R}$ be two hermitian operators and their sum be

$$\hat{S} = \hat{Q} + \hat{R}.$$

Then, for all $f(x)$

$$\langle f|\hat{S}f\rangle = \int f^*\hat{S}f\,dx = \int f^*(\hat{Q}+\hat{R})f\,dx = \int f^*\hat{Q}f\,dx + \int f^*\hat{R}f\,dx$$

$$= \int (\hat{Q}f)^* f\,dx + \int (\hat{R}f)^* f\,dx, \quad \text{(using the hermiticities of } \hat{Q} \text{ and } \hat{R})$$

$$= \int ((\hat{Q}+\hat{R})f)^* f\,dx = \int (\hat{S}f)^* f\,dx = \langle \hat{S}f|f\rangle.$$

Hence, $\hat{S}$ is hermitian.

## Part b

Let,

$$\hat{R} = \alpha\hat{Q}.$$

Then,

$$\langle f|\hat{R}f\rangle = \int f^*\hat{R}f\,dx = \int f^*\alpha\hat{Q}f\,dx = \alpha\int f^*\hat{Q}f\,dx$$

$$= \alpha\int(\hat{Q}f)^*f\,dx, \quad \text{(using the hermiticity of } \hat{Q})$$

$$= \int(\alpha^*\hat{Q}f)^*f\,dx = \int(\alpha\hat{Q}f)^*f\,dx, \quad \text{(if } \alpha \text{ is real).}$$

$$= \int(\hat{R}f)^*f\,dx = \langle\hat{R}f|f\rangle.$$

Hence $\hat{R}$ is hermitian if $\alpha$ is real.

## Part c

Let $\hat{Q}$and $\hat{R}$ be two hermitian operators and let

$$\hat{S} = \hat{Q}\hat{R}.$$

Then,

$$\langle f|\hat{S}f\rangle = \langle f|\hat{Q}\hat{R}f\rangle = \int f^*\hat{Q}\hat{R}f\,dx$$

$$= \int(\hat{Q}f)^*\hat{R}f\,dx = \int(\hat{R}\hat{Q}f)^*f\,dx, \quad \text{(using the hermiticities of } \hat{Q} \text{ and then } \hat{R})$$

$$= \int(\hat{Q}\hat{R}f)^*f\,dx, \quad \text{(if } \hat{Q}\hat{R} = \hat{R}\hat{Q}).$$

$$= \int(\hat{S}f)^*f\,dx = \langle\hat{S}f|f\rangle.$$

Hence $\hat{S}$ is hermitian if $\hat{Q}\hat{R} = \hat{R}\hat{Q}$.

## Part d

$$\langle f|\hat{x}f\rangle = \int f^*xf\,dx = \int(xf)^*f\,dx, \quad \text{(as } x \text{ is real)}$$

$$= \langle\hat{x}f|f\rangle.$$

Hence, $\hat{x}$ is hermitian. Similarly, it can be shown that any function of $\hat{x}$ is also hermitian and hence, $V(x)$ is hermitian.

Equation 3.19 of the textbook shows that $\hat{p}$ is hermitian. Hence, from the result of part c above, $\hat{p}^2$ must also be hermitian. As $(1/2m)$ is real, part b above says that $\hat{p}^2/2m$ is hermitian. But, this is the same as $((-\hbar^2/2m)d^2/dx^2)$. As $V(x)$ is also hermitian, the sum

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x),$$

must also be hermitian as derived in part a.

# Problem 5

## Part a

As $x$ is real,

$$\langle f|\hat{x}g\rangle = \int f^* xg\,dx = \int (xf)^* g\,dx = \langle \hat{x}f|g\rangle.$$

Hence,

$$\hat{x}^\dagger = \hat{x}.$$

$$\langle f|ig\rangle = \int f^* ig\,dx = \int (-if)^* g\,dx = \langle (-if)|g\rangle.$$

Hence,

$$i^\dagger = -i.$$

$$\langle f|\frac{d}{dx}g\rangle = \int_{-\infty}^{\infty} f^* \frac{dg}{dx}\,dx = f^* g|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df^*}{dx} g\,dx = 0 - \int_{-\infty}^{\infty} \frac{df^*}{dx} g\,dx$$

$$= \int_{-\infty}^{\infty} \left(-\frac{d}{dx}f\right)^* g\,dx = \left\langle \left(-\frac{d}{dx}f\right)\middle| g\right\rangle.$$

Hence,

$$\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}$$

## Part b

$$\langle (\hat{Q}\hat{R})^\dagger f|g\rangle = \langle f|(\hat{Q}\hat{R})g\rangle = \langle f|\hat{Q}(\hat{R}g)\rangle = \langle \hat{Q}^\dagger f|\hat{R}g\rangle = \langle \hat{R}^\dagger\hat{Q}^\dagger f|g\rangle.$$

Hence,

$$(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger.$$

$$\langle (\hat{Q}+\hat{R})^\dagger f|g\rangle = \langle f|(\hat{Q}+\hat{R})g\rangle = \langle f|\hat{Q}g\rangle + \langle f|\hat{R}g\rangle = \langle \hat{Q}^\dagger f|g\rangle + \langle \hat{R}^\dagger f|g\rangle = \langle (\hat{Q}^\dagger+\hat{R}^\dagger)f|g\rangle.$$

Hence,

$$(\hat{Q}+\hat{R})^\dagger = \hat{Q}^\dagger + \hat{R}^\dagger.$$

$$\langle (c\hat{Q})^\dagger f|g\rangle = \langle f|(c\hat{Q})g\rangle = \int f^*(c\hat{Q})g\,dx = c\int f^*\hat{Q}g\,dx = c\int (\hat{Q}^\dagger f)^* g\,dx = \int (c^*\hat{Q}^\dagger f)^* g\,dx = \langle c^*\hat{Q}^\dagger f|g\rangle$$

Hence,

$$(c\hat{Q})^\dagger = c^*\hat{Q}^\dagger.$$

## problem 7

### Part a

$$\hat{Q}f = qf, \quad \text{and} \quad \hat{Q}g = qg.$$

So for the linear combination

$$h = \alpha f + \beta g,$$

$$\hat{Q}h = \hat{Q}(\alpha f + \beta g) = \alpha \hat{Q} f + \beta \hat{Q} g = \alpha q f + \beta q g = q(\alpha f + \beta g) = qh.$$

Hence, $h$ is an eigenfunction of $\hat{Q}$ with the eigenvalue $q$.

### Part b

$$\frac{d^2}{dx^2} f = \frac{d^2}{dx^2} \exp(x) = \exp(x) = f.$$

Hence the eigenvalue for $f$ is 1.

$$\frac{d^2}{dx^2} g = \frac{d^2}{dx^2} \exp(-x) = \exp(-x) = g.$$

Hence the eigenvalue for $g$ is also 1. Let the two linear combinations be

$$h_1 = a_1 f + b_1 g, \quad \text{and} \quad h_2 = a_2 f + b_2 g.$$

Then the orthogonality condition is

$$0 = \int_{-1}^{1} h_1^* h_2 \, dx = a_1^* a_2 \int_{-1}^{1} \exp(2x) \, dx + b_1^* b_2 \int_{-1}^{1} \exp(-2x) \, dx + a_1^* b_2 \int_{-1}^{1} dx + b_1^* a_2 \int_{-1}^{1} dx$$
$$= (a_1^* a_2 + b_1^* b_2) \sinh(2) + 2(a_1^* b_2 + b_1^* a_2)$$

This condition can be satisfied an infinite number of ways. One simple choice would be

$$a_1 = a_2 = b_1 = -b_2 = 1.$$

Then,

$$h_1 = f + g, \quad \text{and} \quad h_2 = f - g.$$

## Problem 10

The ground state of the infinite square well is

$$\psi_1 = \sqrt{\frac{2}{a}} \sin(\pi x/a).$$

Operating it with the momentum operator gives

$$\hat{p}\psi_1 = -i\hbar \frac{d\psi_1}{dx} = -i\hbar \sqrt{\frac{2}{a}} \frac{\pi}{a} \cos(\pi x/a).$$

As the above result cannot be written in the form $p\psi_1$, $\psi_1$ is not an eigenfunction of momentum.

## Problem 13

$$\begin{aligned}
\Phi(p,t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi(x,t)\,dx \\
\Phi^*(p,t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx'/\hbar}\Psi^*(x',t)\,dx' \\
i\hbar\frac{\partial \Phi}{\partial p} &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} xe^{-ipx/\hbar}\Psi(x,t)\,dx
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty} \Phi^* \left(i\hbar\frac{\partial}{\partial p}\right)\Phi\,dp &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ipx'/\hbar}\Psi^*(x',t)xe^{-ipx/\hbar}\Psi(x,t)\,dx'\,dx\,dp \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*(x',t)x\Psi(x,t)\,dx'\,dx\int_{-\infty}^{\infty} e^{ipx'/\hbar}e^{-ipx/\hbar}\,dp \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*(x',t)x\Psi(x,t)\,dx'\,dx\int_{-\infty}^{\infty} e^{ip(x'-x)/\hbar}\,dp \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*(x',t)x\Psi(x,t)\,dx'\,dx(2\pi\hbar\delta(x'-x)) \\
&= \int_{-\infty}^{\infty}\Psi^*(x,t)x\Psi(x,t)\,dx = \langle x\rangle.
\end{aligned}$$

## Problem 14

### Part a

$$\begin{aligned}
\left[\hat{A}+\hat{B},\hat{C}\right] &= (\hat{A}+\hat{B})\hat{C} - \hat{C}(\hat{A}+\hat{B}) = \hat{A}\hat{C}+\hat{B}\hat{C}-\hat{C}\hat{A}-\hat{C}\hat{B} \\
&= (\hat{A}\hat{C}-\hat{C}\hat{A}) + (\hat{B}\hat{C}-\hat{C}\hat{B}) = \left[\hat{A},\hat{C}\right]+\left[\hat{B},\hat{C}\right]
\end{aligned}$$

$$\begin{aligned}
\left[\hat{A}\hat{B},\hat{C}\right] &= \hat{A}\hat{B}\hat{C}-\hat{C}\hat{A}\hat{B} = \hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}+\hat{A}\hat{C}\hat{B}-\hat{C}\hat{A}\hat{B} \\
&= \hat{A}(\hat{B}\hat{C}-\hat{C}\hat{B}) + (\hat{A}\hat{C}-\hat{C}\hat{A})\hat{B} = \hat{A}\left[\hat{B},\hat{C}\right]+\left[\hat{A},\hat{C}\right]\hat{B}.
\end{aligned}$$

### Part b

Using an arbitrary test function $f$,

$$[x^n,\hat{p}]f = x^n\hat{p}f - \hat{p}(x^nf) = -i\hbar\left(x^n\frac{df}{dx}-\frac{d}{dx}(x^nf)\right) = -i\hbar\left(x^n\frac{df}{dx}-x^n\frac{df}{dx}-nx^{n-1}f\right) = i\hbar nx^{n-1}f.$$

Hence,

$$[x^n,\hat{p}] = i\hbar nx^{n-1}.$$

## Part c

Let

$$f(x) = \sum_{j=0}^{\infty} a_j x^j.$$

Then, using the results of part a,

$$[f(x), \hat{p}] = \left[\sum_{j=0}^{\infty} a_j x^j, \hat{p}\right] = \sum_{j=0}^{\infty} a_j [x^j, \hat{p}]$$

Then using the result of part b,

$$[f(x), \hat{p}] = i\hbar \sum_{j=0}^{\infty} a_j j x^{j-1} = i\hbar \sum_{j=0}^{\infty} a_j \frac{d}{dx} x^j = i\hbar \frac{d}{dx} \sum_{j=0}^{\infty} a_j x^j = i\hbar \frac{df}{dx}.$$