Solutions
Chapter 2

Problem 1

Part a

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar} = \psi(x)e^{-i(E_0+i\Gamma)t/\hbar} = \psi(x)e^{-iE_0t/\hbar}e^{\Gamma t/\hbar}.$$

Hence,

$$\Psi^*(x,t) = \psi^*(x)e^{iE_0t/\hbar}e^{\Gamma t/\hbar}.$$

So,

$$\int_{-\infty}^{\infty}\Psi^*\Psi dx = \int_{-\infty}^{\infty}\psi^*(x)e^{iE_0t/\hbar}e^{\Gamma t/\hbar}\psi(x)e^{-iE_0t/\hbar}e^{\Gamma t/\hbar}dx = e^{2\Gamma t/\hbar}\int_{-\infty}^{\infty}\psi^*(x)\psi(x)dx$$

As total probability should not change with time, the only way this can be physical is to have $\Gamma = 0$.

Part b

Let ψ be a solution of the time-independent Schrödinger equation with energy E. Then,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi.$$

As both V and E are real, the complex conjugate of this equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi^*}{dx^2} + V\psi^* = E\psi^*.$$

Hence, ψ^* is also a solution of the same Schrödinger equation with the same energy E. As the equation is linear in the wavefunction, the following two linear combinations are also solutions.

$$\psi_r = \psi + \psi^* \quad \text{and} \quad \psi_i = i(\psi - \psi^*).$$

That is,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_r}{dx^2} + V\psi_r = E\psi_r,$$

and

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_i}{dx^2} + V\psi_i = E\psi_i.$$

If ψ is not real, then ψ^* is another distinct solution with the same energy. But then they can both be replaced by the two real solutions ψ_r and ψ_i. That these are real is easily verified as follows.

$$\psi_r^* = (\psi + \psi^*)^* = \psi^* + \psi = \psi_r,$$

and,

$$\psi_i^* = (i(\psi - \psi^*))^* = -i\psi^* + i\psi = \psi_i.$$

Part c

Replacing x by $-x$ in the time-independent Schrödinger equation gives,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(-x)}{dx^2} + V(-x)\psi(-x) = E\psi(-x).$$

If $V(x)$ is even, then $V(-x) = V(x)$. Hence,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(-x)}{dx^2} + V(x)\psi(-x) = E\psi(-x).$$

So, $\psi(x)$ and $\psi(-x)$ are both solutions of the same equations. Hence, from the linearity of the equation, we conclude that the following are both solutions as well.

$$\psi_e(x) = \psi(x) + \psi(-x) \quad \text{and} \quad \psi_o(x) = \psi(x) - \psi(-x).$$

Then the pair of solutions $\psi(x)$ and $\psi(-x)$ can be replaced by the pair $\psi_e(x)$ and $\psi_o(x)$. $\psi_e(x)$ is even as

$$\psi_e(-x) = \psi(-x) + \psi(-(-x)) = \psi_e(x).$$

$\psi_o(x)$ is odd as

$$\psi_o(-x) = \psi(-x) - \psi(-(-x)) = -\psi_o(x).$$

Problem 2

The Schrödinger equation can be written in the following form.

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi.$$

If $E < V_{min}$, then $[V(x) - E]$ is positive for all x. Hence, ψ and its second derivative have the same sign for all x. Now consider the following cases of x dependence of ψ starting at any arbitrary x – say $x = x_0$ and moving in the positive x direction.

1. ψ is positive and $d\psi/dx$ is positive. Then $d^2\psi/dx^2$ is also positive from above observation. This will make ψ increase in the positive x direction until it becomes infinity at $x = \infty$.

2. ψ is positive and $d\psi/dx$ is negative. Then $d^2\psi/dx^2$ is also positive from above observation. This will make ψ decrease and $d\psi/dx$ increase in the positive x direction as long as ψ and hence, $d^2\psi/dx^2$ remain positive. This will continue until $d\psi/dx$ becomes zero at a minimum point and proceeds to increase in the positive x direction. Then ψ will also increase beyond the minimum point until it becomes infinity at $x = \infty$. However, if $d\psi/dx$ fails to become zero before ψ becomes zero, then ψ and hence, $d^2\psi/dx^2$ will become negative and item 3 will become applicable.

3. ψ is negative and $d\psi/dx$ is negative. Then $d^2\psi/dx^2$ is also negative from above observation. This will make ψ decrease in the positive x direction until it becomes negative infinity at $x = -\infty$.

4. ψ is negative and $d\psi/dx$ is positive. Then $d^2\psi/dx^2$ is also negative from above observation. This will make ψ increase and $d\psi/dx$ decrease in the positive x direction as long as ψ and hence, $d^2\psi/dx^2$ remain negative. This will continue until $d\psi/dx$ becomes zero at a maximum point and proceeds to decrease in the positive x direction. Then ψ will also decrease beyond the maximum point until it becomes negative infinity at $x = -\infty$. However, if $d\psi/dx$ fails to become zero before ψ becomes zero, then ψ and hence, $d^2\psi/dx^2$ will become positive and item 1 will become applicable.

The above items cover all cases and they all lead to ψ becoming infinite (positive or negative) at $x = \infty$. Hence, normalization will not be possible.

Problem 4

$$\langle x \rangle = \frac{2}{a}\int_0^a x \sin^2(n\pi x/a)dx = \frac{1}{a}\int_0^a x(1-\cos(2n\pi x/a))dx = \frac{a}{2}.$$

$$\langle x^2 \rangle = \frac{2}{a}\int_0^a x^2 \sin^2(n\pi x/a)dx = \frac{1}{a}\int_0^a x^2(1-\cos(2n\pi x/a))dx = a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right).$$

$$\langle p \rangle = \frac{2}{a}(-i\hbar)\int_0^a \sin(n\pi x/a)\frac{d}{dx}\sin(n\pi x/a)dx = \frac{-2i\hbar n\pi}{a^2}\int_0^a \sin(n\pi x/a)\cos(n\pi x/a)dx = 0.$$

$$\langle p^2 \rangle = \frac{2}{a}(-i\hbar)^2\int_0^a \sin(n\pi x/a)\frac{d^2}{dx^2}\sin(n\pi x/a)dx = \frac{2n^2\pi^2\hbar^2}{a^3}\int_0^a \sin^2(n\pi x/a)dx = \frac{n^2\pi^2\hbar^2}{a^2}.$$

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) - \frac{a^2}{4} = a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2} - \frac{1}{4}\right) = \frac{a^2}{4}\left(\frac{1}{3} - \frac{2}{n^2\pi^2}\right)$$

$$\sigma_x = \frac{a}{2}\sqrt{\left(\frac{1}{3} - \frac{2}{n^2\pi^2}\right)}$$

$$\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 = \frac{n^2\pi^2\hbar^2}{a^2}$$

$$\sigma_p = \frac{n\pi\hbar}{a}$$

$$\sigma_x\sigma_p = \frac{n\pi\hbar}{2}\sqrt{\left(\frac{1}{3} - \frac{2}{n^2\pi^2}\right)} = \frac{\hbar}{2}\sqrt{\left(\frac{n^2\pi^2}{3} - 2\right)}$$

This is the smallest for $n = 1$ and it is

$$\sigma_x\sigma_p = \frac{\hbar}{2}\sqrt{\left(\frac{\pi^2}{3} - 2\right)} > \frac{\hbar}{2}$$

Problem 15

The recursion formula is,

$$a_{j+2} = \frac{-2(n-j)}{(j+1)(j+2)} a_j.$$

For $H_5(\xi)$,

$$a_3 = \frac{-2(5-1)}{2 \times 3} a_1 = -4a_1/3, \quad a_5 = \frac{-2(5-3)}{4 \times 5} a_3 = 4a_1/15.$$

Hence,

$$H_5(\xi) = a_1(\xi - 4\xi^3/3 + 4\xi^5/15)$$

To keep with convention,

$$4a_1/15 = 2^5, \quad \text{Hence,} \quad a_1 = 120.$$

Thus,

$$H_5(\xi) = 32\xi^5 - 160\xi^3 + 120\xi.$$

For $H_6(\xi)$,

$$a_2 = \frac{-2(6-0)}{1 \times 2} a_0 = -6a_0, \quad a_4 = \frac{-2(6-2)}{3 \times 4} a_2 = 4a_0, \quad a_6 = \frac{-2(6-4)}{5 \times 6} a_4 = -8a_0/15.$$

Hence,

$$H_6(\xi) = a_0(1 - 6\xi^2 + 4\xi^4 - 8\xi^6/15).$$

To keep with convention,

$$-8a_0/15 = 2^6, \quad \text{Hence,} \quad a_0 = -120.$$

Thus,

$$H_6(\xi) = 64\xi^6 - 480\xi^4 + 720\xi^2 - 120.$$

Problem 20

Part a

$$1 = \int \Psi^* \Psi dx = A^2 \int_{-\infty}^{\infty} e^{-2a|x|} dx = 2A^2 \int_0^{\infty} e^{-2ax} dx = A^2/a.$$

Hence,

$$A = \sqrt{a}.$$

Part b

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx = \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-ikx} dx$$

$$= \sqrt{\frac{a}{2\pi}} \left[\int_{-\infty}^{0} e^{ax} e^{-ikx} dx + \int_0^{\infty} e^{-ax} e^{-ikx} dx \right] = \sqrt{\frac{a}{2\pi}} \left[\frac{1}{a-ik} + \frac{1}{a+ik} \right] = \sqrt{\frac{2a^3}{\pi}} \frac{1}{a^2+k^2}.$$

Part c

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i(kx-\hbar k^2 t/2m)}dk = \frac{a^{3/2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{i(kx-\hbar k^2 t/2m)}}{a^2+k^2}dk.$$

Problem 21

Part a

$$1 = \int \Psi^*\Psi dx = A^2\int_{-\infty}^{\infty} e^{-2ax^2}dx = A^2\sqrt{\frac{\pi}{2a}}.$$

Hence,

$$A = \left(\frac{2a}{\pi}\right)^{1/4}$$

Part b

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}dx = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ax^2}e^{-ikx}dx = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a(x^2+ikx/a)}dx$$
$$= \frac{A}{\sqrt{2\pi}}e^{-k^2/(4a)}\int_{-\infty}^{\infty}e^{-a(x^2+ikx/a-k^2/(4a^2))}dx = \frac{A}{\sqrt{2\pi}}e^{-k^2/(4a)}\int_{-\infty}^{\infty}e^{-a(x+ik/(2a))^2}dx$$
$$= \frac{A}{\sqrt{2\pi}}e^{-k^2/(4a)}\sqrt{\frac{\pi}{a}} = \frac{A}{\sqrt{2a}}e^{-k^2/(4a)} = \frac{1}{(2\pi a)^{1/4}}e^{-k^2/(4a)}$$
$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i(kx-\hbar k^2 t/2m)}dk = \frac{1}{(8\pi^3 a)^{1/4}}\int_{-\infty}^{\infty}e^{-k^2/(4a)}e^{i(kx-\hbar k^2 t/2m)}dk$$
$$= \frac{1}{(8\pi^3 a)^{1/4}}\int_{-\infty}^{\infty}e^{-\alpha k^2}e^{ikx}dk$$

where,

$$\alpha = \frac{1}{4a} + \frac{i\hbar t}{2m}$$

With that,

$$\Psi(x,t) = \frac{1}{(8\pi^3 a)^{1/4}}\int_{-\infty}^{\infty}e^{-\alpha(k^2-ikx/\alpha)}dk = \frac{e^{-x^2/4\alpha}}{(8\pi^3 a)^{1/4}}\int_{-\infty}^{\infty}e^{-\alpha(k^2-ikx/\alpha-x^2/4\alpha^2)}dk$$
$$= \frac{e^{-x^2/4\alpha}}{(8\pi^3 a)^{1/4}}\int_{-\infty}^{\infty}e^{-\alpha(k-ix/2\alpha)^2}dk = \frac{e^{-x^2/4\alpha}}{(8\pi^3 a)^{1/4}}\sqrt{\frac{\pi}{\alpha}}.$$

Note that,

$$4\alpha = \frac{1}{a} + \frac{2i\hbar t}{m} = \frac{1}{a}\left(1 + \frac{2ia\hbar t}{m}\right) = \frac{\gamma^2}{a},$$

as γ is defined to be,

$$\gamma = \sqrt{\left(1 + \frac{2ia\hbar t}{m}\right)}$$

Then,

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-ax^2/\gamma^2}.$$

Part c

$$\Psi^* = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\gamma^*} e^{-ax^2/\gamma^{*2}}.$$

$$|\Psi(x,t)|^2 = \Psi^*\Psi = \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\gamma^*\gamma} e^{-ax^2(1/\gamma^{*2}+1/\gamma^2)}$$

$$\gamma^* = \sqrt{\left(1 - \frac{2ia\hbar t}{m}\right)}, \quad \gamma^{*2} = \left(1 - \frac{2ia\hbar t}{m}\right).$$

$$\gamma^*\gamma = \sqrt{\left(1 + \frac{4a^2\hbar^2 t^2}{m^2}\right)} = \frac{\sqrt{a}}{w} \quad \text{as} \quad w = \sqrt{a/(1 + 4a^2\hbar^2t^2/m^2)}$$

$$1/\gamma^{*2} + 1/\gamma^2 = \frac{\gamma^2 + \gamma^{*2}}{\gamma^2\gamma^{*2}} = \frac{2}{\left(1 + \frac{4a^2\hbar^2t^2}{m^2}\right)} = \frac{2w^2}{a}$$

Hence,

$$|\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}} w e^{-2w^2x^2}$$

Part d

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx = \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x e^{-2w^2x^2} dx = 0.$$

$$\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x} dx = i\hbar(2a/\gamma^2) \int_{-\infty}^{\infty} x|\Psi|^2 dx = 0.$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2 dx = \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x^2 e^{-2w^2x^2} dx = \frac{1}{4w^2}.$$

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial x^2} dx$$

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{\partial}{\partial x}\left(\left(\frac{2a}{\pi}\right)^{1/4}\frac{2ax}{\gamma^3}e^{-ax^2/\gamma^2}\right) = -\left(\frac{2a}{\pi}\right)^{1/4}\frac{2a}{\gamma^3}\frac{\partial}{\partial x}\left(xe^{-ax^2/\gamma^2}\right)$$

$$= -\left(\frac{2a}{\pi}\right)^{1/4}\frac{2a}{\gamma^3}(1-2ax^2/\gamma^2)e^{-ax^2/\gamma^2} = -\frac{2a}{\gamma^2}(1-2ax^2/\gamma^2)\Psi$$

Hence,

$$\langle p^2\rangle = \hbar^2\frac{2a}{\gamma^2}\int_{-\infty}^{\infty}(1-2ax^2/\gamma^2)|\Psi|^2dx = \hbar^2\frac{2a}{\gamma^2}\left(1-(2a/\gamma^2)\int_{-\infty}^{\infty}x^2|\Psi|^2dx\right)$$

$$= \hbar^2\frac{2a}{\gamma^2}\left(1-(2a/\gamma^2)\frac{1}{4w^2}\right) = \hbar^2\frac{2a}{\gamma^2}\left(1-\frac{a}{2\gamma^2w^2}\right) = \hbar^2\frac{2a}{\gamma^2}\left(1-\frac{a(1+(2a\hbar t/m)^2)}{2(1+2ia\hbar t/m)a}\right)$$

$$= \hbar^2\frac{2a}{\gamma^2}\left(\frac{2(1+2ia\hbar t/m)-(1+(2a\hbar t/m)^2)}{2(1+2ia\hbar t/m)}\right) = \hbar^2\frac{2a}{\gamma^2}\left(\frac{1+4ia\hbar t/m-(2a\hbar t/m)^2)}{2(1+2ia\hbar t/m)}\right)$$

$$= \hbar^2\frac{2a}{\gamma^2}\left(\frac{(1+2ia\hbar t/m)^2}{2(1+2ia\hbar t/m)}\right) = \hbar^2\frac{2a}{\gamma^2}\left(\frac{\gamma^4}{2\gamma^2}\right) = a\hbar^2.$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1}{2w}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{a}\hbar.$$

Hence,

$$\sigma_x\sigma_p = \frac{\sqrt{a}\hbar}{2w} = \frac{\hbar}{2}\sqrt{(1+4a^2\hbar^2t^2/m^2)}$$

This has the minimum value at $t=0$ and increases with t.

Problem 34

Part a

For $x \le 0$, the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

The solution of this is

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad \text{where} \quad k = \sqrt{\frac{2mE}{\hbar^2}}.$$

For $x > 0$, the Schrödinger equation is

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (V_0 - E)\psi.$$

The equation is written in this form as $E < V_0$. The solution of this is

$$\psi(x) = Ce^{-\kappa x} + De^{\kappa x} \quad \text{where} \quad \kappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}.$$

As the wavefunction cannot go to ∞ at $x = \infty$, we need to impose the condition $D = 0$. So,

$$\psi(x) = Ce^{-\kappa x} \quad \text{for} \quad x > 0.$$

The continuity condition on ψ at $x = 0$ gives

$$A + B = C.$$

The continuity condition on $d\psi/dx$ at $x = 0$, gives

$$ikA - ikB = -\kappa C.$$

Eliminating C between these equations gives

$$ik(A - B) = -\kappa(A + B).$$

Hence,

$$B = \frac{ik + \kappa}{ik - \kappa} A,$$

and the reflection coefficient is,

$$R = \left|\frac{B}{A}\right|^2 = \left|\frac{ik + \kappa}{ik - \kappa}\right|^2 = 1.$$

Hence, everything is reflected and there is no transmission.

Part b

For $x \leq 0$, the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

The solution of this is

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad \text{where} \quad k = \sqrt{\frac{2mE}{\hbar^2}}.$$

For $x > 0$, the Schrödinger equation is

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = -(E - V_0)\psi.$$

The equation is written in this form as $E > V_0$. The solution of this is

$$\psi(x) = Ce^{iKx} + De^{-iKx} \quad \text{where} \quad K = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}.$$

As there can be no wave traveling in the negative x direction in this region, we need to impose the condition $D = 0$. So,

$$\psi(x) = Ce^{iKx} \quad \text{for} \quad x > 0.$$

The continuity condition on ψ at $x = 0$ gives

$$A + B = C.$$

The continuity condition on $d\psi/dx$ at $x = 0$, gives

$$ikA - ikB = iKC.$$

Eliminating C between these equations gives

$$ik(A - B) = iK(A + B).$$

Hence,

$$B = \frac{k - K}{k + K}A,$$

and the reflection coefficient is,

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{k - K}{k + K}\right)^2.$$

which is less than 1 as long as $E > V_0 > 0$.

Part c

The definition of particle current (problem 1.14) is

$$J(x,t) = \frac{i\hbar}{2m}\left(\frac{\partial\Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{i\hbar}{2m}\left(\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right)$$

For $x < 0$, the incident particle wavefunction is

$$\psi_{in} = Ae^{ikx}.$$

So, the incident particle current is

$$J_{in} = \frac{i\hbar}{2m}(-2ik|A|^2) = \frac{\hbar k}{m}|A|^2$$

For $x > 0$, the transmitted particle wavefunction is

$$\psi_{out} = Ce^{iKx}.$$

So, the transmitted particle current is

$$J_{out} = \frac{i\hbar}{2m}(-2iK|C|^2) = \frac{\hbar K}{m}|C|^2$$

Hence, the transmission coefficient is

$$T = \frac{J_{out}}{J_{in}} = \frac{K}{k}\frac{|C|^2}{|A|^2} = \sqrt{\frac{E - V_0}{E}}\frac{|C|^2}{|A|^2}.$$

Part d

From part b, we see that

$$C = A + B = A + \frac{k-K}{k+K}A = \frac{2k}{k+K}A.$$

Hence, from the results of part c,

$$T = \frac{K}{k}\frac{4k^2}{(k+K)^2} = \frac{4kK}{(k+K)^2}.$$

Then,

$$R + T = \left(\frac{k-K}{k+K}\right)^2 + \frac{4kK}{(k+K)^2} = \frac{k^2 - 2kK + K^2 + 4kK}{(k+K)^2} = \frac{(k+K)^2}{(k+K)^2} = 1.$$