Solutions

Chapter 1

Problem 1

Part a

$$\langle j \rangle = \frac{\sum jN(j)}{N} = 14 \times 1/14 + 15 \times 1/14 + 16 \times 3/14 + 22 \times 2/14 + 24 \times 2/14 + 25 \times 5/14 = 21.$$

So,

$$\langle j \rangle^2 = 21^2 = 441$$

$$\langle j^2 \rangle = \frac{\sum j^2 N(j)}{N} = 14^2 \times 1/14 + 15^2 \times 1/14 + 16^2 \times 3/14 + 22^2 \times 2/14 + 24^2 \times 2/14 + 25^2 \times 5/14 = 459.6.$$

Part b

$$\Delta j_{14} = 14 - 21 = -7, \quad \Delta j_{15} = 15 - 21 = -6, \quad \Delta j_{16} = 16 - 21 = -5, \quad \Delta j_{22} = 22 - 21 = 1,$$
$$\Delta j_{24} = 24 - 21 = 3, \quad \Delta j_{25} = 25 - 21 = 4.$$

Hence,

$$\sigma^2 = \langle (\Delta j)^2 \rangle = [7^2 + 6^2 + 5^2 \times 3 + 1^2 \times 2 + 3^2 \times 2 + 4^2 \times 5]/14 = 18.6$$

and,

$$\sigma = 4.31$$

Part c

$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} = 4.31$$

Problem 3

Part a

$$1 = A \int_{-\infty}^{\infty} e^{-\lambda (x-a)^2} dx = A\sqrt{\pi/\lambda}$$

Hence,

$$A = \sqrt{\lambda/\pi}.$$

Part b

$$\langle x\rangle = \int_{-\infty}^{\infty} x\rho\,dx = A\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}\,dx = A\int_{-\infty}^{\infty}(y+a)e^{-\lambda y^2}\,dy$$

$$= A\left(\int_{-\infty}^{\infty} ye^{-\lambda y^2}\,dy + a\int_{-\infty}^{\infty} e^{-\lambda y^2}\,dy\right) = 0 + Aa\int_{-\infty}^{\infty} e^{-\lambda y^2}\,dy = a.$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2\rho\,dx = A\int_{-\infty}^{\infty} x^2e^{-\lambda(x-a)^2}\,dx = A\int_{-\infty}^{\infty}(y+a)^2e^{-\lambda y^2}\,dy$$

$$= A\left(\int_{-\infty}^{\infty} y^2e^{-\lambda y^2}\,dy + 2a\int_{-\infty}^{\infty} ye^{-\lambda y^2}\,dy + a^2\int_{-\infty}^{\infty} e^{-\lambda y^2}\,dy\right) = A\left(\frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}} + 0 + a^2\sqrt{\pi/\lambda}\right)$$

$$= \frac{1}{2\lambda} + a^2$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{1}{2\lambda}$$

$$\sigma = 1/\sqrt{2\lambda}$$

Problem 5

Part a

$$1 = \int \Psi^*\Psi dx = A^2\int_{-\infty}^{\infty} e^{-2\lambda|x|}dx = 2A^2\int_0^{\infty} e^{-2\lambda x}dx = A^2/\lambda.$$

Hence,

$$A = \sqrt{\lambda}$$

Part b

$$\langle x\rangle = A^2\int_{-\infty}^{\infty} xe^{-2\lambda|x|}dx = 0.$$

$$\langle x^2\rangle = A^2\int_{-\infty}^{\infty} x^2e^{-2\lambda|x|}dx = 2A^2\int_0^{\infty} x^2e^{-2\lambda x}dx = 2A^2\frac{1}{4\lambda^3} = \frac{1}{2\lambda^2}.$$

Part c

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2}$$

Hence,

$$\sigma = \frac{1}{\sqrt{2}\lambda}$$

Probability outside range $(-\sigma, +\sigma)$ is

$$P_o = 2\int_{\sigma}^{\infty} A^2 e^{-2\lambda x} dx = 2A^2 \frac{1}{2\lambda} e^{-2\lambda\sigma} = e^{-\sqrt{2}}$$

Problem 7

$$\frac{d\langle p \rangle}{dt} = -i\hbar \frac{d}{dt}\int \Psi^* \frac{\partial \Psi}{\partial x} dx = -i\hbar \int \left(\frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x}\frac{\partial \Psi}{\partial t} \right) dx$$

Then, using the forms of the Schrödinger equation in equations 1.23 and 1.24 of the book,

$$\frac{d\langle p \rangle}{dt} = -i\hbar \int \left(\left(-\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V \Psi^* \right) \frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x} \left(\frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V \Psi \right) \right) dx$$

$$= -i\hbar \int \left(-\frac{i\hbar}{2m} \left(\frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial^3 \Psi}{\partial x^3} \right) + \frac{i}{\hbar} \left(V\Psi^* \frac{\partial \Psi}{\partial x} - V\Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial V}{\partial x}\Psi \right) \right) dx$$

Using the fact that the wavefunction and its derivatives vanish at infinity, and integrating one of the two terms by parts twice, it is seen that

$$\int \left(\frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial^3 \Psi}{\partial x^3} \right) dx = 0.$$

Hence,

$$\frac{d\langle p \rangle}{dt} = -\int \Psi^* \frac{\partial V}{\partial x} \Psi dx = \left\langle -\frac{\partial V}{\partial x} \right\rangle.$$

Problem 9

Part a

$$1 = \int \Psi^* \Psi dx = A^2 \int_{-\infty}^{\infty} e^{-2amx^2/\hbar} dx = A^2 \sqrt{\frac{\pi\hbar}{2am}}$$

Hence,

$$A = \left(\frac{2am}{\pi\hbar} \right)^{1/4}$$

Part b

$$\frac{\partial \Psi}{\partial t} = -ia\Psi.$$

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{2am}{\hbar}\frac{\partial}{\partial x}(x\Psi) = -\frac{2am}{\hbar}\left((\Psi - \frac{2am}{\hbar}x^2\Psi\right) = -\frac{2am}{\hbar}\left(1 - \frac{2am}{\hbar}x^2\right)\Psi$$

Inserting this into the Schrödinger equation gives

$$i\hbar(-ia)\Psi = \frac{\hbar^2}{2m}\frac{2am}{\hbar}\left(1 - \frac{2am}{\hbar}x^2\right)\Psi + V\Psi.$$

Hence,

$$V = 2ma^2x^2.$$

Part c

$$\langle x \rangle = \int \Psi^* x \Psi dx = A^2 \int_{-\infty}^{\infty} x e^{-2amx^2/\hbar} dx = 0.$$

$$\langle x^2 \rangle = \int \Psi^* x^2 \Psi dx = A^2 \int_{-\infty}^{\infty} x^2 e^{-2amx^2/\hbar} dx = \frac{A^2}{2}\sqrt{\frac{\pi\hbar^3}{8a^3m^3}}$$

$$= \frac{1}{2}\sqrt{\frac{2am}{\pi\hbar}}\sqrt{\frac{\pi\hbar^3}{8a^3m^3}} = \frac{\hbar}{4am}$$

$$\langle p \rangle = -i\hbar \int \Psi^* \frac{\partial \Psi}{\partial x} dx = i\hbar A^2 \frac{2am}{\hbar}\int_{-\infty}^{\infty} x e^{-2amx^2/\hbar} dx = 0.$$

$$\langle p^2 \rangle = -\hbar^2 \int \Psi^* \frac{\partial^2 \Psi}{\partial x^2} dx$$

Using earlier computation of $\partial^2\Psi/\partial x^2$,

$$\langle p^2 \rangle = -\hbar^2 \int \Psi^* \left(-\frac{2am}{\hbar}\left(1 - \frac{2am}{\hbar}x^2\right)\Psi\right) dx$$

$$= 2am\hbar\left(\int \Psi^*\Psi dx - \frac{2am}{\hbar}\int \Psi^* x^2 \Psi dx\right) = 2am\hbar\left(1 - \frac{2am}{\hbar}\langle x^2 \rangle\right)$$

Using earlier computation of $\langle x^2 \rangle$,

$$\langle p^2 \rangle = am\hbar$$

Part d

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{4am}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{am\hbar}$$

Hence,

$$\sigma_x \sigma_p = \hbar/2.$$

This is the minimum uncertainty product allowed by the uncertainty principle.

Problem 14

Part a

$$P_{ab} = \int_a^b \Psi^* \Psi dx.$$

So,

$$\frac{dP_{ab}}{dt} = \int_a^b \left(\frac{\partial \Psi^*}{dt} \Psi + \Psi^* \frac{\partial \Psi}{dt} \right) dx$$

Then, using the forms of the Schrödinger equation in equations 1.23 and 1.24 of the book,

$$\frac{dP_{ab}}{dt} = \int_a^b \left(\left(-\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V \Psi^* \right) \Psi + \Psi^* \left(\frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V \Psi \right) \right) dx$$

The terms including V cancel out leaving the following.

$$\frac{dP_{ab}}{dt} = \frac{i\hbar}{2m} \int_a^b \left(-\frac{\partial^2 \Psi^*}{\partial x^2} \Psi + \Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \right) dx$$

Then, using the method of integration by parts,

$$\frac{dP_{ab}}{dt} = \frac{i\hbar}{2m} \int_a^b \left(-\frac{\partial}{\partial x} \left(\frac{\partial \Psi^*}{\partial x} \Psi \right) + \frac{\partial \Psi^*}{\partial x} \frac{\partial \Psi}{\partial x} + \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} \right) - \frac{\partial \Psi^*}{\partial x} \frac{\partial \Psi}{\partial x} \right) dx$$

$$= \frac{i\hbar}{2m} \int_a^b \left(-\frac{\partial}{\partial x} \left(\frac{\partial \Psi^*}{\partial x} \Psi \right) + \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} \right) \right) dx = \frac{i\hbar}{2m} \left(-\frac{\partial \Psi^*}{\partial x} \Psi + \Psi^* \frac{\partial \Psi}{\partial x} \right) \Big|_a^b$$

$$= J(a,t) - J(b,t).$$

Part b

$$J(x,t) = \frac{i\hbar}{2m} \left(\frac{\partial \Psi^*}{\partial x} \Psi - \Psi^* \frac{\partial \Psi}{\partial x} \right) = \frac{i\hbar}{2m} \left(-\frac{2amx}{\hbar} \Psi^* \Psi + \frac{2amx}{\hbar} \Psi^* \Psi \right) = 0.$$