Solutions

Chapter 2

Problem 2

First consider $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ to be parallel. Hence, we can choose them both to be in the x direction:

$$E_x = E_0 f(z - ct), \quad B_x = B_0 f(z - ct).$$

Then the integral on the right side of the third Maxwell equation is,

$$\int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \int (B_x \hat{\mathbf{i}}) \cdot (dA\hat{\mathbf{j}}) = 0,$$

as $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$. Then, applying the third Maxwell equation as done in the text, will give,

$$E_0 = 0.$$

A similar argument for the fourth Maxwell equation will give,

$$B_0 = 0.$$

Hence, there will be no wave.

Next, let the electric and magnetic field directions remain perpendicular, but choose the direction of wave propagation to be the x direction. Then,

$$E_x = E_0 f(x - ct), \quad B_y = B_0 f(x - ct).$$

And, following the method in the text, we get the loop integral,

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = (E_x(z + dz) - E_x(z))dx = \frac{\partial E_x}{\partial z} dz\, dx = 0.$$

This is because E_x does not depend on z in this case. Inserting this into the third Maxwell equation gives,

$$B_0 = 0.$$

Similarly, using the fourth Maxwell equation will give,

$$E_0 = 0.$$

Hence, there will be no wave.

Problem 4

$$\begin{aligned}
\sum_{\mu=1}^{4} x_\mu'^2 &= (x_1 \cos\theta - x_2 \sin\theta)^2 + (x_1 \sin\theta + x_2 \cos\theta)^2 + x_3^2 + x_4^2 \\
&= x_1^2 \cos^2\theta + x_2^2 \sin^2\theta - 2x_1 x_2 \cos\theta \sin\theta + x_1^2 \sin^2\theta + x_2^2 \cos^2\theta + 2x_1 x_2 \cos\theta \sin\theta + x_3^2 + x_4^2 \\
&= +x_1^2 + x_2^2 + x_3^2 + x_4^2 = \sum_{\mu=1}^{4} x_\mu^2
\end{aligned}$$

Problem 5

$$\begin{aligned}
\sum_{\mu=1}^{4} x'^2_\mu &= (x_1 \cos\phi - x_4 \sin\phi)^2 + x_2^2 + x_3^2 + (x_1 \sin\phi + x_4 \cos\phi)^2 \\
&= x_1^2 \cos^2\phi + x_4^2 \sin^2\phi - 2x_1x_4 \cos\phi \sin\phi + x_2^2 + x_3^2 + x_1^2 \sin^2\phi + x_4^2 \cos^2\phi + 2x_1x_4 \cos\phi \sin\phi \\
&= +x_1^2 + x_2^2 + x_3^2 + x_4^2 = \sum_{\mu=1}^{4} x_\mu^2
\end{aligned}$$

Problem 8

Rotation about the z axis by an angle θ is

$$R(\hat{\mathbf{k}}, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For a velocity v along the x axis, the equivalent rotation by an imaginary angle is given by ϕ where

$$v = ic \tan\phi$$

Then the corresponding transformation matrix is

$$B(\hat{\mathbf{i}}, \phi) = \begin{pmatrix} \cos\phi & 0 & 0 & -\sin\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\phi & 0 & 0 & \cos\phi \end{pmatrix}$$

So, a rotation about the z axis followed by a Lorentz transformation due to a velocity in the x direction is

$$\begin{aligned}
L &= B(\hat{\mathbf{i}}, \phi) R(\hat{\mathbf{k}}, \theta) \\
&= \begin{pmatrix} \cos\phi & 0 & 0 & -\sin\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\phi & 0 & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\cos\phi & 0 & -\sin\phi \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cos\theta\sin\phi & -\sin\theta\sin\phi & 0 & \cos\phi \end{pmatrix}
\end{aligned}$$

Problem 9

Lorentz transformation due to a velocity v in the y direction is ($v = ic\tan\phi$)

$$B(\hat{\mathbf{j}}, \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{pmatrix}$$

Problem 10

The Lorentz transformation due to a velocity v_1 along the x axis is ($v_1 = ic\tan\phi_1$)

$$B(\hat{\mathbf{i}}, \phi_1) = \begin{pmatrix} \cos\phi_1 & 0 & 0 & -\sin\phi_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\phi_1 & 0 & 0 & \cos\phi_1 \end{pmatrix}$$

The Lorentz transformation due to a velocity v_2 along the y axis is ($v_2 = ic\tan\phi_2$)

$$B(\hat{\mathbf{j}}, \phi_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_2 & 0 & -\sin\phi_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\phi_2 & 0 & \cos\phi_2 \end{pmatrix}$$

So, a Lorentz transformation representing $B(\hat{\mathbf{i}}, \phi_1)$ followed by $B(\hat{\mathbf{j}}, \phi_2)$ is given by

$$\begin{aligned} L &= B(\hat{\mathbf{j}}, \phi_2)B(\hat{\mathbf{i}}, \phi_1) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_2 & 0 & -\sin\phi_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\phi_2 & 0 & \cos\phi_2 \end{pmatrix} \begin{pmatrix} \cos\phi_1 & 0 & 0 & -\sin\phi_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\phi_1 & 0 & 0 & \cos\phi_1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi_1 & 0 & 0 & -\sin\phi_1 \\ -\sin\phi_1\sin\phi_2 & \cos\phi_2 & 0 & -\cos\phi_1\sin\phi_2 \\ 0 & 0 & 1 & 0 \\ \sin\phi_1\cos\phi_2 & \sin\phi_2 & 0 & \cos\phi_1\cos\phi_2 \end{pmatrix} \end{aligned}$$

Problem 14

For the case of positive v_1 and v_2, let $\beta_1 = v_1/c$ and $\beta_2 = v_2/c$. Then, using the hint,

$$0 < (1-\beta_1)(1-\beta_2) = 1 - \beta_1 - \beta_2 + \beta_1\beta_2$$

Hence,

$$\beta_1 + \beta_2 < 1 + \beta_1\beta_2$$

Dividing throughout by $(1 + \beta_1\beta_2)$ gives

$$\frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} < 1$$

Multiplying throughout by c gives

$$\frac{v_1 + v_2}{1 + v_1 v_2/c^2} < c$$

Problem 15

The fractional contraction is

$$f = \frac{L_0 - L}{L_0} = 1 - \sqrt{1 - v^2/c^2}$$

Using the binomial expansion

$$(1 + x)^n = 1 + nx + n(n + 1)x^2/2 + \cdots$$

upto the first order term (as v/c is very small)

$$f = 1 - (1 - v^2/c^2)^{1/2} \simeq 1 - (1 - (1/2)v^2/c^2) = (1/2)v^2/c^2 = 4.7 \times 10^{-15}$$

Problem 16

The amount by which it is longer is

$$\Delta T = T - T_0 = (\gamma - 1)T_0 = ((1 - v^2/c^2)^{-1/2} - 1)T_0 \simeq (1 + (1/2)v^2/c^2 - 1)T_0 = 4.7 \times 10^{-15} \text{ hr.}$$

as T_0 is 1 hour.

Problem 22

$$T' = \gamma(T - vX/c^2) \geq \gamma(T - vTc/c^2)$$

as $X \leq Tc$. Then,

$$T' \geq \gamma T(1 - v/c) \geq 0.$$

Problem 23

As $(\Delta s)^2$ is an invariant, its sign is preserved in a Lorentz transformation. Then, "likenesses" must also be preserved under a Lorentz transformation as the sign of $(\Delta s)^2$ completely determines the "likeness" – negative for time-like, zero for light-like and positive for space-like.

Problem 24

When measured from the side (transverse Doppler), considering the two successive shifts gives

$$\nu = \nu_0\sqrt{1 - v^2/c^2}\sqrt{1 - v^2/c^2} = \nu_0(1 - v^2/c^2)$$

So the decrease in frequency is

$$\nu_0 - \nu = \nu_0 v^2/c^2 = 5.6 \times 10^{-4} \text{ sec}^{-1}$$

When measured from behind (longitudinal Doppler), considering the two successive shifts gives

$$\nu = \nu_0\sqrt{\frac{1 - v/c}{1 + v/c}}\sqrt{\frac{1 - v/c}{1 + v/c}} = \nu_0\frac{1 - v/c}{1 + v/c} \simeq \nu_0(1 - v/c)(1 - v/c) \simeq \nu_0(1 - 2v/c)$$

So the decrease in frequency is

$$\nu_0 - \nu = 2\nu_0 v/c = 6.7 \times 10^3 \text{ sec}^{-1}$$

Problem 29

Let the magnitude of the electron and proton charge be e, the electron mass be m, the electron speed be v and the orbit radius be r. Then, as the electrostatic force on the electron is also its centripetal force:

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2}$$

Hence, the kinetic energy is

$$K = mv^2/2 = \frac{e^2}{8\pi\epsilon_0 r}$$

The potential energy is

$$U = -\frac{e^2}{4\pi\epsilon_0 r}$$

Hence, the total energy is

$$E = K + U = -\frac{e^2}{8\pi\epsilon_0 r}$$

Here, $e = 1.6 \times 10^{-19}$ C, $\epsilon_0 = 8.9 \times 10^{-12}$ F/m and $r = 5.3 \times 10^{-11}$ m. So,

$$E = -2.2 \times 10^{-18} \text{ J}$$

The mass equivalent of this is

$$\Delta m = E/c^2 = 2.4 \times 10^{-35} \text{ kg}$$

As the electron mass is negligible compared to the proton mass m_p, the fractional difference of the hydrogen atom mass and the mass of its constituents is

$$\frac{\Delta m}{m_p} = 1.4 \times 10^{-8}$$

Problem 30

The mass-shell condition is,

$$\sum_{\mu=1}^{4} p_\mu^2 = p_1^2 + p_2^2 + p_3^2 + p_4^2 = -m_0^2 c^2.$$

As $p_4 = imc = iE/c$, this gives,

$$p_1^2 + p_2^2 + p_3^2 - E^2/c^2 = -m_0^2 c^2.$$

As $p^2 = p_1^2 + p_2^2 + p_3^2$ by definition,

$$p^2 - E^2/c^2 = -m_0^2 c^2.$$

Multiplying by c^2 and rearranging terms will give,

$$E^2 - p^2 c^2 = m_0^2 c^4.$$

Problem 31

As the rest-mass of a photon is zero, the mass-shell condition reduces to,

$$E^2 - p^2 c^2 = 0.$$

Hence,

$$p = E/c.$$

Also, for a photon,

$$E = h\nu.$$

Hence,

$$p = h\nu/c = h/\lambda,$$

noting that, for any wave, frequency ν, wavelength λ and speed c are related by,

$$c = \nu\lambda.$$

Problem 32

The collision being along one direction, momentum conservation gives only one equation:

$$h/\lambda = -h/\lambda' + m_0\gamma v \tag{1}$$

where λ' is the wavelength of the photon after collision, m_0 the rest mass of the electron and v its speed after collision. The energy conservation equation gives

$$hc/\lambda + m_0c^2 = hc/\lambda' + m_0\gamma c^2 \tag{2}$$

The above two equations can be rewritten as

$$h/\lambda + h/\lambda' = m_0\gamma v \tag{3}$$

$$h/\lambda - h/\lambda' + m_0c = m_0\gamma c \tag{4}$$

Squaring both sides of equations 3 and 4 and subtracting gives

$$(h/\lambda - h/\lambda' + m_0c)^2 - (h/\lambda + h/\lambda')^2 = m_0^2\gamma^2(c^2 - v^2) \tag{5}$$

Further simplifying this gives

$$m_0^2c^2 + 2m_0ch\frac{\lambda' - \lambda}{\lambda'\lambda} - 4\frac{h^2}{\lambda'\lambda} = m_0^2c^2 \tag{6}$$

This leads to

$$\lambda' = \lambda + \frac{2h}{m_0c} \tag{7}$$

To find v one may solve for γ from equation 4.

$$\gamma = \frac{h(\lambda' - \lambda)}{m_0c\lambda'\lambda} + 1 \tag{8}$$

Using equation 7 this becomes

$$\gamma = \frac{2h^2}{m_0c\lambda(m_0c\lambda + 2h)} + 1 \tag{9}$$

and using the definition of γ we know

$$v = c\sqrt{1 - 1/\gamma^2} \tag{10}$$