Solutions
Chapter 5

Problem 2

The general solution from Ex. 5.2 is

$$\begin{aligned} y(t) &= C_1\cos(\omega t) + C_2\sin(\omega t) + (E/B)t + C_3. \\ z(t) &= C_2\cos(\omega t) - C_1\sin(\omega t) + C_4 \end{aligned}$$

Part a

In this part $y(0) = 0$, $z(0) = 0$, $\dot{y}(0) = E/B$ and $\dot{z}(0) = 0$. Hence,

$$\begin{aligned} 0 &= C_1 + C_3. \\ 0 &= C_2 + C_4 \\ E/B &= C_2\omega + (E/B) \\ 0 &= -C_1\omega \end{aligned}$$

The solution for these 4 equations gives $C_1 = C_2 = C_3 = C_4 = 0$. Hence,

$$\begin{aligned} y(t) &= (E/B)t. \\ z(t) &= 0 \end{aligned}$$

Part b

In this part $y(0) = 0$, $z(0) = 0$, $\dot{y}(0) = E/(2B)$ and $\dot{z}(0) = 0$. Hence,

$$\begin{aligned} 0 &= C_1 + C_3. \\ 0 &= C_2 + C_4 \\ E/(2B) &= C_2\omega + (E/B) \\ 0 &= -C_1\omega \end{aligned}$$

The solution for these 4 equations gives $C_1 = C_3 = 0$, and $C_2 = -C_4 = -E/(2\omega B)$. Hence,

$$\begin{aligned} y(t) &= -\frac{E}{2\omega B}\sin(\omega t) + \frac{E}{B}t. \\ z(t) &= -\frac{E}{2\omega B}\cos(\omega t) + \frac{E}{2\omega B} \end{aligned}$$

Part c

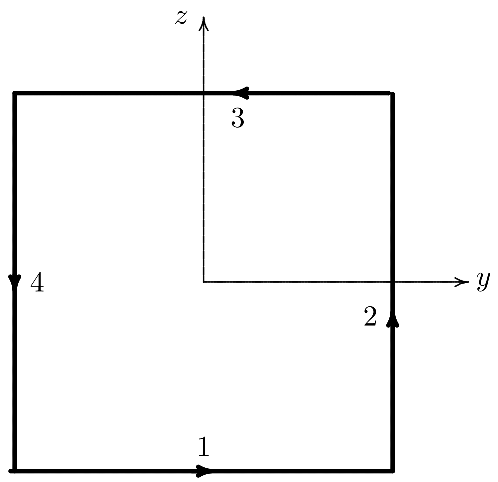
In this part $y(0) = 0$, $z(0) = 0$, $\dot{y}(0) = E/B$ and $\dot{z}(0) = E/B$. Hence,

$$\begin{aligned} 0 &= C_1 + C_3. \\ 0 &= C_2 + C_4 \\ E/B &= C_2\omega + (E/B) \\ E/B &= -C_1\omega \end{aligned}$$

The solution for these 4 equations gives $C_1 = -C_3 = -E/(\omega B)$, and $C_2 = C_4 = 0$. Hence,

$$\begin{aligned} y(t) &= -\frac{E}{\omega B}\cos(\omega t) + \frac{E}{B}t + \frac{E}{\omega B}. \\ z(t) &= \frac{E}{\omega B}\sin(\omega t) \end{aligned}$$

Problem 4



The four sides of the square are labeled 1, 2, 3, and 4. So, the magnetic force can be written as

$$\begin{aligned} \vec{\mathbf{F}} = I\int d\vec{\mathbf{l}} \times \vec{\mathbf{B}} &= I\left(\int_1 d\vec{\mathbf{l}} \times \vec{\mathbf{B}} + \int_2 d\vec{\mathbf{l}} \times \vec{\mathbf{B}} + \int_3 d\vec{\mathbf{l}} \times \vec{\mathbf{B}} + \int_4 d\vec{\mathbf{l}} \times \vec{\mathbf{B}}\right) \\ &= I\left(\int_1 dy\hat{\mathbf{y}} \times (kz\hat{\mathbf{x}}) + \int_2 dz\hat{\mathbf{z}} \times (kz\hat{\mathbf{x}}) + \int_3 dy\hat{\mathbf{y}} \times (kz\hat{\mathbf{x}}) + \int_4 dz\hat{\mathbf{z}} \times (kz\hat{\mathbf{x}})\right) \\ &= I\left(\int_{-a/2}^{a/2} kz(-\hat{\mathbf{z}})dy + \int_{-a/2}^{a/2} kz\hat{\mathbf{y}}dz + \int_{a/2}^{-a/2} kz(-\hat{\mathbf{z}})dy + \int_{a/2}^{-a/2} kz\hat{\mathbf{y}}dz\right) \end{aligned}$$

Note that on side 1, $z = -a/2$ and on side 3 $z = a/2$. Hence,

$$\begin{aligned} &= Ik\left(-\hat{\mathbf{z}}(-a/2)\int_{-a/2}^{a/2} dy + \hat{\mathbf{y}}\int_{-a/2}^{a/2} zdz - \hat{\mathbf{z}}(a/2)\int_{a/2}^{-a/2} dy + \hat{\mathbf{y}}\int_{a/2}^{-a/2} zdz\right) \\ &= Ik\left(\hat{\mathbf{z}}a^2/2 + 0 + \hat{\mathbf{z}}a^2/2 + 0\right) = Ika^2\hat{\mathbf{z}}. \end{aligned}$$

Problem 9

$$\vec{\mathbf{B}}(\vec{\mathbf{r}}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\mathbf{l}}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

Part a

The straight parts have $\vec{\mathbf{l}}' \times \hat{\mathscr{r}} = 0$. So, only the curved parts contribute to the field. Choosing the center of the arc as origin of the coordinate system makes $\vec{\mathbf{r}} = 0$ and $\mathscr{r} = -\vec{\mathbf{r}}'$. So, $\mathscr{r}$ points towards the center. and the direction of $d\vec{\mathbf{l}}' \times \hat{\mathscr{r}}$ for the inner arc is out of the page (right-hand-rule). Hence, the magnetic field produced by it is out of the page. Also $|d\vec{\mathbf{l}}' \times \hat{\mathscr{r}}| = dl' |\hat{\mathscr{r}}| \sin(90°) = dl'$. So the magnitude of the magnetic field is

$$B_a = \frac{\mu_0 I}{4\pi} \int_0^{\pi a/2} \frac{dl'}{a^2} = \frac{\mu_0 I}{8a}.$$

The computation of the magnetic field due to the outer arc is similar. As the current direction is opposite, the direction of $d\vec{\mathbf{l}}'$ is opposite, making the magnetic field into the page. The corresponding magnitude of the field is

$$B_b = \frac{\mu_0 I}{4\pi} \int_0^{\pi b/2} \frac{dl'}{b^2} = \frac{\mu_0 I}{8b}.$$

Hence, the net magnitude is

$$B = B_a - B_b = \frac{\mu_0 I}{8} \left(\frac{1}{a} - \frac{1}{b} \right),$$

and the direction is out of the page.

Part b

Using the right-hand-rule, the direction of the field is into the page. The computation of the magnetic field due to the semicircular part of the wire is similar to the computation in part (a) with the integration upper limit being πR. It is as follows.

$$B_R = \frac{\mu_0 I}{4\pi} \int_0^{\pi R} \frac{dl'}{R^2} = \frac{\mu_0 I}{4R}.$$

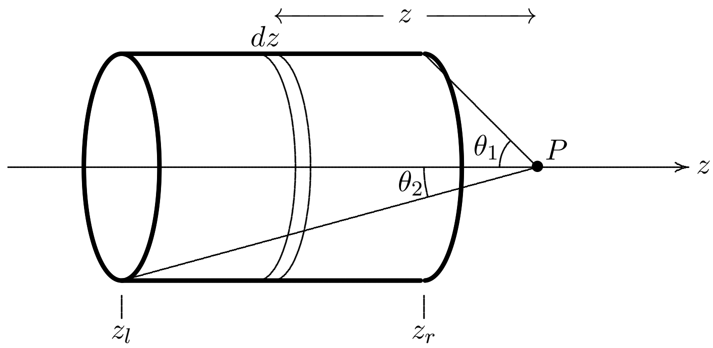
The top and bottom straight sections of the wire have the same contribution and its computation is like that in example 5.5 of the book with $\theta_1 = 0$ and $\theta_2 = \pi/2$. Hence, the corresponding magnetic fields B_t and B_b are

$$B_t = B_b = \frac{\mu_0 I}{4\pi R} (\sin(\pi/2) - \sin 0) = \frac{\mu_0 I}{4\pi R}.$$

So, the net magnetic field is

$$B = B_R + B_t + B_b = \frac{\mu_0 I}{4R} + \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{4\pi R} = \frac{\mu_0 I}{2R}(1/2 + 1/\pi).$$

Problem 11



Consider the axis of the solenoid to be the z axis with its origin at the target point P. Approximating the packing of the wires to be continuous, the current in a thin ring of thickness dz is $In\,dz$. So the magnetic field due to it is given by equation 5.41 (example 5.6) to be

$$dB = \frac{\mu_0 In\,dz a^2}{2(a^2+z^2)^{3/2}},$$

where z is the coordinate of the thin ring. Hence, if z_l and z_r are the z coordinates of the left and the right ends of the solenoid,

$$B = \int_{z_l}^{z_r} \frac{\mu_0 In\,dz a^2}{2(a^2+z^2)^{3/2}} = \frac{\mu_0 Ina^2}{2}\int_{z_l}^{z_r}\frac{dz}{(a^2+z^2)^{3/2}}$$

Substituting $z = a\tan\phi$ gives

$$B = \frac{\mu_0 In}{2}\int_{\phi_l}^{\phi_r}\cos\phi\,d\phi = \frac{\mu_0 In}{2}(\sin\phi_r - \sin\phi_l) = \frac{\mu_0 In}{2}\left(\frac{z_r}{\sqrt{a^2+z_r^2}} - \frac{z_l}{\sqrt{a^2+z_l^2}}\right)$$

Using the angles θ_1 and θ_2 defined in the problem,

$$\cos\theta_1 = -\frac{z_r}{\sqrt{a^2+z_r^2}}, \quad \cos\theta_2 = -\frac{z_l}{\sqrt{a^2+z_l^2}}.$$

Hence,

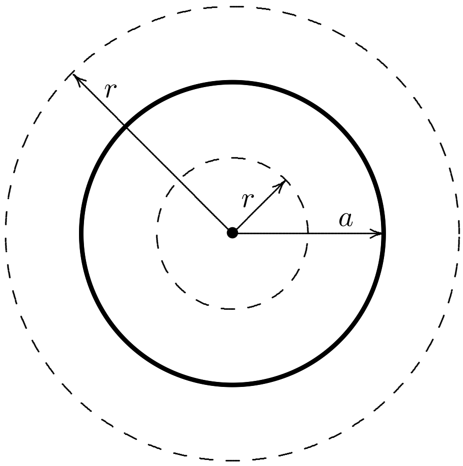
$$B = \frac{\mu_0 In}{2}(\cos\theta_2 - \cos\theta_1).$$

For the infinitely long solenoid $\theta_1 = \pi$ and $\theta_2 = 0$. Hence, the magnetic field for that is

$$B = \mu_0 nI.$$

Problem 14

The following is the cross-section of the wire and the two dashed circles show the amperean loops for computation of the field outside ($r > a$) and inside ($r < a$) the wire. The current is out of the page.



For all cases, if the integral is done in the counterclockwise direction,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \oint B\, dl = B \oint dl = B(2\pi r).$$

As B is constant over a circular loop due to symmetry.

Part a

For the magnetic field B_i inside the wire the inside amperean loop ($r < a$) is used. The current within it is zero as the current is only on the surface of the wire. Hence,

$$B_i(2\pi r) = 0.$$

Hence,

$$B_i = 0.$$

For the magnetic field B_o outside the wire the outside amperean loop ($r > a$) is used. The current within it is I. Hence,

$$B_o(2\pi r) = \mu_0 I.$$

Hence,

$$B_o = \frac{\mu_0 I}{2\pi r}.$$

Part b

For the magnetic field B_i inside the wire the inside amperean loop ($r < a$) is used. The current within it is given by

$$I_{enc} = \int J\, da',$$

where J is the current density, da' is the area element and the integration region is inside the amperean loop. As J is given to be proportional to s the distance from the center,

$$J = ks,$$

where k is a constant. Due to circular symmetry da' can be seen as thin rings of radius s and thickness ds The area of such a ring is $da' = (2\pi s)ds$. Hence,

$$I_{enc} = \int_0^r ks(2\pi s)ds = 2\pi k r^3/3.$$

Similar computation for the total current gives

$$I = \int_0^a ks(2\pi s)ds = 2\pi k a^3/3.$$

This gives,

$$I_{enc} = Ir^3/a^3$$

Hence,

$$B_i(2\pi r) = \mu_0 I r^3/a^3,$$

and,

$$B_i = \frac{\mu_0 I r^2}{2\pi a^3}.$$

For the magnetic field B_o outside the wire the outside amperean loop ($r > a$) is used. The current within it is I. Hence,

$$B_o(2\pi r) = \mu_0 I.$$

Hence,

$$B_o = \frac{\mu_0 I}{2\pi r}.$$

Problem 17

Part a

The moving top sheet is like a current. Its current per unit length, like in example 5.8, is

$$K = \sigma v.$$

The direction is towards right. Using the result of example 5.8 gives the magnetic field produced by the sheet to have a magnitude of

$$B_1 = \mu_0 K/2 = \mu_0 \sigma v/2.$$

The direction is out of the page above the sheet and into the page below it.

The moving bottom sheet, having a negative charge of the same magnitude produces a magnetic field of the same magnitude as above. But the direction is reversed – into the page above and out of the page below. So the net magnetic field above the top sheet is zero as the contributions of the two surfaces cancel. Same happens below the bottom sheet. However, between the two sheets, both contributions are into the page and they add to give

$$B = 2B_1 = \mu_0 \sigma v.$$

Part b

The force is due to the magnetic field $\vec{\mathbf{B}}_b$ of the bottom sheet. The force on a charge dQ on the top sheet is then given by

$$d\vec{\mathbf{F}} = dQ\vec{\mathbf{v}} \times \vec{\mathbf{B}}_b.$$

The force per unit area is

$$\frac{d\vec{\mathbf{F}}}{da} = \frac{dQ}{da}\vec{\mathbf{v}} \times \vec{\mathbf{B}}_b = \sigma \vec{\mathbf{v}} \times \vec{\mathbf{B}}_b.$$

The direction is upwards and the magnitude is

$$\frac{dF}{da} = \sigma|\vec{\mathbf{v}} \times \vec{\mathbf{B}}_b| = \sigma v B_1 = \mu_0 \sigma^2 v^2/2.$$

Part c

The electric field due to the bottom surface is downwards and of magnitude

$$E_1 = \frac{\sigma}{2\epsilon_0}.$$

The magnitude of the force on a charge dQ due to this field is

$$dF_e = dQ\, E_1 = dQ \frac{\sigma}{2\epsilon_0}.$$

The force per unit area is

$$\frac{dF_e}{da} = \frac{dQ}{da}\frac{\sigma}{2\epsilon_0} = \frac{\sigma^2}{2\epsilon_0}.$$

If the electric and magnetic forces are equal in magnitude,

$$\mu_0 \sigma^2 v^2/2 = \frac{\sigma^2}{2\epsilon_0},$$

which gives,

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

This is the speed of light in vacuum.