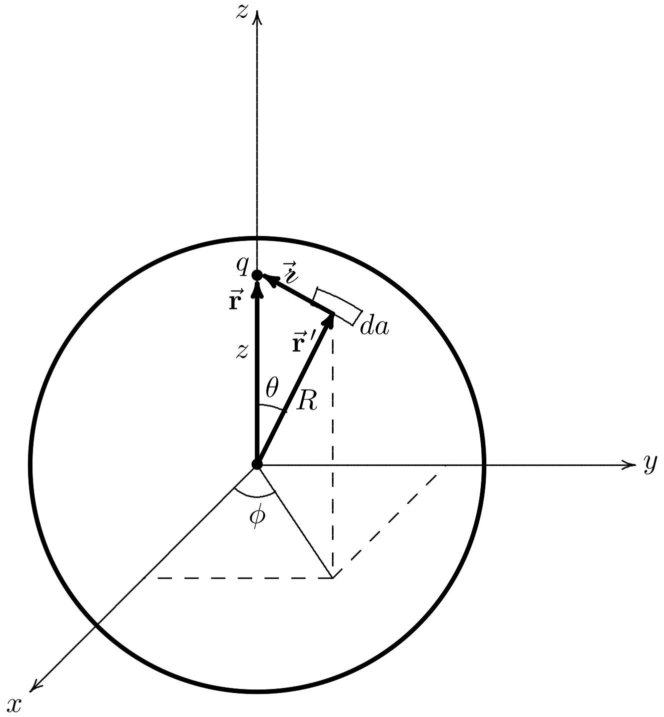
Solutions

Chapter 3

Problem 1



Consider a charge q placed inside the sphere of radius R at a distance z from the origin along the z axis ($z < R$). Then,

$$\vec{\mathbf{r}} = z\hat{\mathbf{z}}.$$

Converting rectangular to spherical polar coordinates for the surface element shown,

$$\vec{\mathbf{r}}\,' = R\sin\theta\cos\phi\hat{\mathbf{x}} + R\sin\theta\sin\phi\hat{\mathbf{y}} + R\cos\theta\hat{\mathbf{z}},$$

$$\vec{\mathbf{r}} - \vec{\mathbf{r}}\,' = -R\sin\theta\cos\phi\hat{\mathbf{x}} - R\sin\theta\sin\phi\hat{\mathbf{y}} + (z - R\cos\theta)\hat{\mathbf{z}}.$$

Hence,

$$(R^2\sin^2\theta\cos^2\phi + R^2\sin^2\theta\sin^2\phi + z^2 + R^2\cos^2\theta - 2Rz\cos\theta)^{1/2} = (R^2 + z^2 - 2Rz\cos\theta)^{1/2},$$

$$V_{\text{av}} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int\frac{da}{ } = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{R^2\sin\theta d\theta d\phi}{(R^2 + z^2 - 2Rz\cos\theta)^{1/2}}.$$

The ϕ integral gives

$$\int_0^{2\pi} d\phi = 2\pi.$$

Using the substitution $u = (R^2 + z^2 - 2Rz\cos\theta)^{1/2}$, the θ integral gives

$$\int_0^\pi \frac{\sin\theta\, d\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{1/2}} = \frac{1}{Rz}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right] = 2/R,$$

as $z < R$.

Hence,

$$V\text{av} = \frac{q}{4\pi\epsilon_0 R}$$

So, irrespective of the actual position of the charge, the above equation gives the average potential due to it on the sphere as long as it is inside the sphere. That makes the total contribution of all charges inside the sphere to the average to be

$$V\text{av} = \frac{Q_{enc}}{4\pi\epsilon_0 R}.$$

The contribution of external charges has already been found to be V_{center}. So the average potential on the sphere due to all charges (inside and outside) is

$$V\text{av} = V_{center} + \frac{Q_{enc}}{4\pi\epsilon_0 R}$$

Problem 13

In this case,

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)dy = \frac{2}{a}\left[\int_0^{a/2} V_0(y)\sin(n\pi y/a)dy + \int_{a/2}^a V_0(y)\sin(n\pi y/a)dy,\right]$$

$$= \frac{2}{a}\left[V_0\int_0^{a/2}\sin(n\pi y/a)dy - V_0\int_{a/2}^a \sin(n\pi y/a)dy,\right] = \frac{2V_0}{n\pi}[1 - 2\cos(n\pi/2) + \cos(n\pi)]$$

$$= \begin{cases} \frac{8V_0}{n\pi}, & \text{if } n = 2, 6, 10, \ldots \\ 0, & \text{otherwise .} \end{cases}$$

Hence,

$$V(x,y) = \frac{8V_0}{\pi}\sum_{n=2,6,10,\ldots} \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a).$$

Problem 19

Using trigonometry,

$$V_0 = k\cos(3\theta) = 4k\cos^3\theta - 3k\cos\theta.$$

As this has only odd powers of $\cos\theta$ up to a power of 3, it can be written as a linear combination of $P_1(\cos\theta)$ and $P_3(\cos\theta)$. Hence, let

$$4kx^3 - 3kx = akP_1(x) + bkP_3(x),$$

where $x = \cos\theta$ and a and b are constants yet to be determined. As x^3 appears only in P_3 with a coefficient of $5/2$,

$$4 = 5b/2, \quad \text{and hence,} \quad b = 8/5.$$

Then, from the last equation,

$$4x^3 - 3x = aP_1(x) + \frac{8}{5}P_3(x) = ax + 4x^3 - \frac{12}{5}x.$$

Hence,

$$a = -3/5,$$

and

$$V_0 = \frac{8k}{5}P_3(\cos\theta) - \frac{3k}{5}P_1(\cos\theta).$$

This means

$$V(R,\theta) = V_0 = \frac{k}{5}(8P_3(\cos\theta) - 3P_1(\cos\theta)) \tag{1}$$

So, the potential inside ($r < R$) is

$$V(r,\theta) = \frac{k}{5}\left(\frac{8r^3}{R^3}P_3(\cos\theta) - \frac{3r}{R}P_1(\cos\theta)\right).$$

Writing it in terms of $\cos\theta$ gives

$$V(r,\theta) = \frac{k}{5R}\left(\frac{20r^3}{R^2}\cos^3\theta - \frac{12r^3}{R^2}\cos\theta - 3r\cos\theta\right).$$

Using equation 1 for points outside ($r > R$) gives

$$V(r,\theta) = \frac{k}{5}\left(\frac{8R^4}{r^4}P_3(\cos\theta) - \frac{3R^2}{r^2}P_1(\cos\theta)\right).$$

Writing it in terms of $\cos\theta$ gives

$$V(r,\theta) = \frac{kR^2}{5}\left(\frac{20R^2}{r^4}\cos^3\theta - \frac{12R^2}{r^4}\cos\theta - \frac{3}{r^2}\cos\theta\right).$$