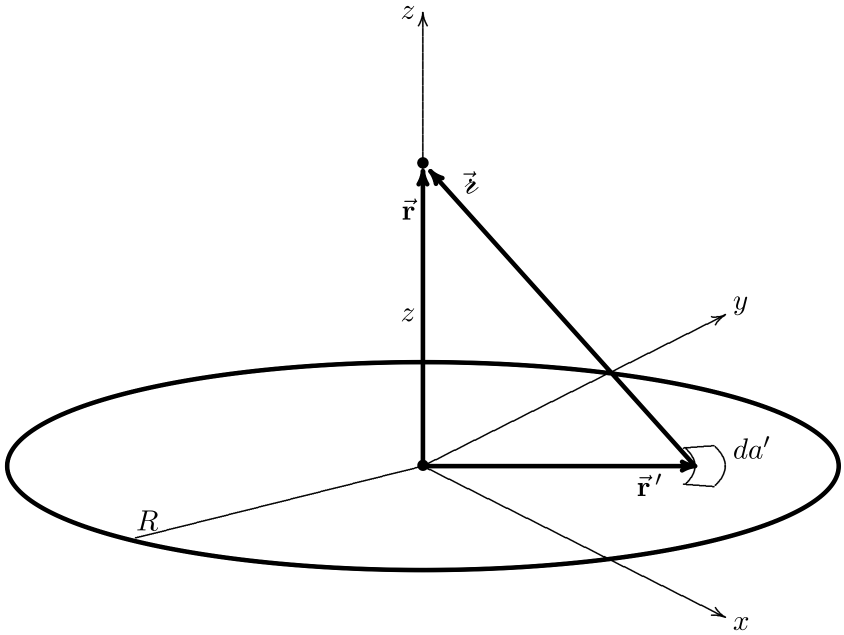
Solutions
Chapter 2

Problem 6



$$\vec{\mathbf{r}} = z\hat{\mathbf{z}}, \quad \vec{\mathbf{r}}\,' = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}, \quad \vec{\mathscr{r}} = \vec{\mathbf{r}} - \vec{\mathbf{r}}\,' = z\hat{\mathbf{z}} - x\hat{\mathbf{x}} - y\hat{\mathbf{y}}$$

$$\mathscr{r} = (x^2 + y^2 + z^2)^{1/2}, \quad \hat{\mathscr{r}} = \vec{\mathscr{r}}/\mathscr{r}$$

Hence,

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(z\hat{\mathbf{z}} - x\hat{\mathbf{x}} - y\hat{\mathbf{y}})}{(x^2 + y^2 + z^2)^{3/2}} da'$$

In cylindrical polar coordinates (ρ, ϕ, z), $x = \rho\cos\phi, y = \rho\sin\phi$ and $da' = \rho\, d\rho\, d\phi$. Hence,

$$\vec{\mathbf{E}} = \frac{\sigma}{4\pi\epsilon_0}\left[-\hat{\mathbf{x}}\int_0^{2\pi}\int_0^R \frac{\rho\cos\phi\,\rho\,d\rho\,d\phi}{(\rho^2+z^2)^{3/2}} - \hat{\mathbf{y}}\int_0^{2\pi}\int_0^R \frac{\rho\sin\phi\,\rho\,d\rho\,d\phi}{(\rho^2+z^2)^{3/2}} + \hat{\mathbf{z}}\int_0^{2\pi}\int_0^R \frac{z\,\rho\,d\rho\,d\phi}{(\rho^2+z^2)^{3/2}}\right].$$

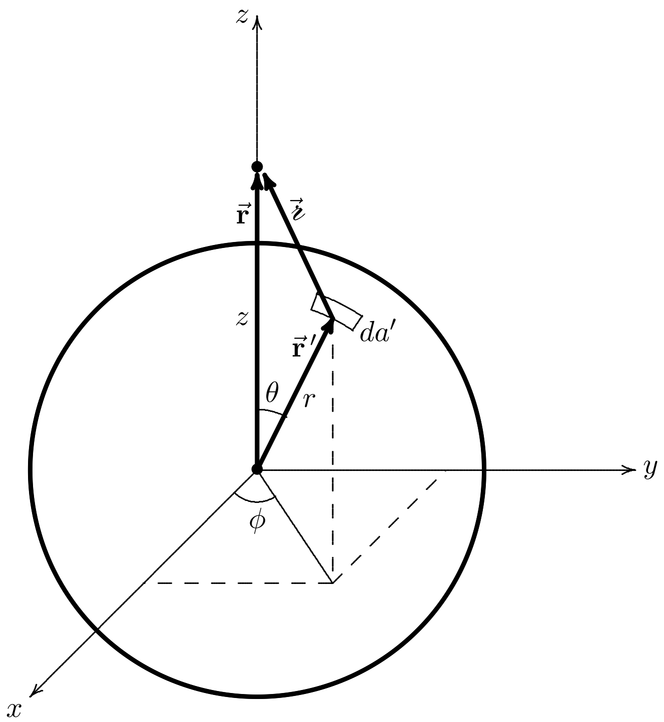
The x and the y components vanish as

$$\int_0^{2\pi}\cos\phi\,d\phi = \int_0^{2\pi}\sin\phi\,d\phi = 0.$$

Hence,

$$\vec{\mathbf{E}} = \frac{\sigma\,z\,\hat{\mathbf{z}}}{4\pi\epsilon_0}\left[\int_0^{2\pi}\int_0^R \frac{\rho\,d\rho\,d\phi}{(\rho^2+z^2)^{3/2}}\right] = \frac{2\pi\sigma\,z\,\hat{\mathbf{z}}}{4\pi\epsilon_0}\left[\int_0^R \frac{\rho\,d\rho}{(\rho^2+z^2)^{3/2}}\right] = \frac{\sigma\,z\,\hat{\mathbf{z}}}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{(R^2+z^2)^{1/2}}\right].$$

Problem 7



$$\vec{\mathbf{r}} = z\hat{\mathbf{z}}.$$

Converting rectangular to spherical polar coordinates on the surface of the sphere gives,

$$\vec{\mathbf{r}}' = R\sin\theta\cos\phi\hat{\mathbf{x}} + R\sin\theta\sin\phi\hat{\mathbf{y}} + R\cos\theta\hat{\mathbf{z}},$$

$$\vec{\mathscr{r}} = \vec{\mathbf{r}} - \vec{\mathbf{r}}' = -R\sin\theta\cos\phi\hat{\mathbf{x}} - R\sin\theta\sin\phi\hat{\mathbf{y}} + (z - R\cos\theta)\hat{\mathbf{z}}.$$

Hence,

$$\mathscr{r} = (R^2\sin^2\theta\cos^2\phi + R^2\sin^2\theta\sin^2\phi + z^2 + R^2\cos^2\theta - 2Rz\cos\theta)^{1/2} = (R^2 + z^2 - 2Rz\cos\theta)^{1/2},$$

and,

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{-R\sin\theta\cos\phi\hat{\mathbf{x}} - R\sin\theta\sin\phi\hat{\mathbf{y}} + (z - R\cos\theta)\hat{\mathbf{z}}}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}$$

Using this in the general formula for $\vec{\mathbf{E}}$ gives,

$$\vec{\mathbf{E}} = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\left[\frac{-R\sin\theta\cos\phi\hat{\mathbf{x}} - R\sin\theta\sin\phi\hat{\mathbf{y}} + (z - R\cos\theta)\hat{\mathbf{z}}}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}\right]R^2\sin\theta\,d\theta\,d\phi,$$

using the spherical polar form $da' = R^2\sin\theta\,d\theta\,d\phi$. Computing the ϕ integral makes the x and y components vanish and then,

$$\vec{\mathbf{E}} = \frac{2\pi R^2\sigma\hat{\mathbf{z}}}{4\pi\epsilon_0}\int_0^{\pi}\left[\frac{(z - R\cos\theta)}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}\right]\sin\theta\,d\theta.$$

Various integration tricks can be used to compute this integral. One of them comes from the observation that,

$$-\frac{d}{dz}\frac{1}{(R^2+z^2-2Rz\cos\theta)^{1/2}} = \frac{(z-R\cos\theta)}{(R^2+z^2-2Rz\cos\theta)^{3/2}}.$$

This gives,

$$\vec{\mathbf{E}} = -\frac{R^2\sigma\hat{\mathbf{z}}}{2\epsilon_0}\frac{d}{dz}\int_0^\pi \left[\frac{1}{(R^2+z^2-2Rz\cos\theta)^{1/2}}\right]\sin\theta\, d\theta.$$

Substituting $u = (R^2+z^2-2Rz\cos\theta)$ allows the computation of this integral. It amounts to

$$\vec{\mathbf{E}} = -\frac{R^2\sigma\hat{\mathbf{z}}}{2\epsilon_0}\frac{1}{R}\frac{d}{dz}\left[\frac{1}{z}\left.(R^2+z^2-2Rz\cos\theta)^{1/2}\right|_0^\pi\right]$$

$$= \frac{R\sigma\hat{\mathbf{z}}}{2\epsilon_0}\frac{d}{dz}\left[\frac{1}{z}\left[(R^2+z^2-2Rz)^{1/2}-(R^2+z^2+2Rz)^{1/2}\right]\right]$$

$$= \frac{R\sigma\hat{\mathbf{z}}}{2\epsilon_0}\frac{d}{dz}\left[\frac{1}{z}\left[\sqrt{(R-z)^2}-\sqrt{(R+z)^2}\right]\right].$$

At this stage, it is important to note that the square root is always the positive one. So,

$$\sqrt{(R-z)^2} = \begin{cases} R-z & \text{if } R>z, \\ z-R & \text{if } R<z. \end{cases}$$

Hence, for $z<R$ (inside the sphere),

$$\vec{\mathbf{E}} = \frac{R\sigma\hat{\mathbf{z}}}{2\epsilon_0}\frac{d}{dz}\left[\frac{1}{z}[-2z]\right] = 0,$$

and for $z>R$ (outside the sphere),

$$\vec{\mathbf{E}} = \frac{R\sigma\hat{\mathbf{z}}}{2\epsilon_0}\frac{d}{dz}\left[\frac{1}{z}[-2R]\right] = \frac{R^2\sigma\hat{\mathbf{z}}}{\epsilon_0 z^2} = \frac{4\pi R^2\sigma\hat{\mathbf{z}}}{4\pi\epsilon_0 z^2} = \frac{q\hat{\mathbf{z}}}{4\pi\epsilon_0 z^2},$$

as $4\pi R^2$ is the surface area of the sphere and the total charge is $q = 4\pi R^2\sigma$. This shows the unexpected but well-known result that the electric field inside the sphere is zero and outside the sphere the same as if the total charge were a point charge at the center.

Problem 8

For points outside the solid sphere, consider the sphere to be made up of thin concentric shells. Each shell contributes an electric field that is due to its charge as if it were at the center. So, the net electric field would be as if the total charge q were all at the center. Hence,

$$\vec{\mathbf{E}} = \frac{q\hat{\mathbf{z}}}{4\pi\epsilon_0 z^2}, \quad \text{for } z>R.$$

For a point at some distance z such that $z<R$, we can still visualize the sphere as composed of infinitesimally thin spherical shells. The shells that have radii greater than z do not contribute

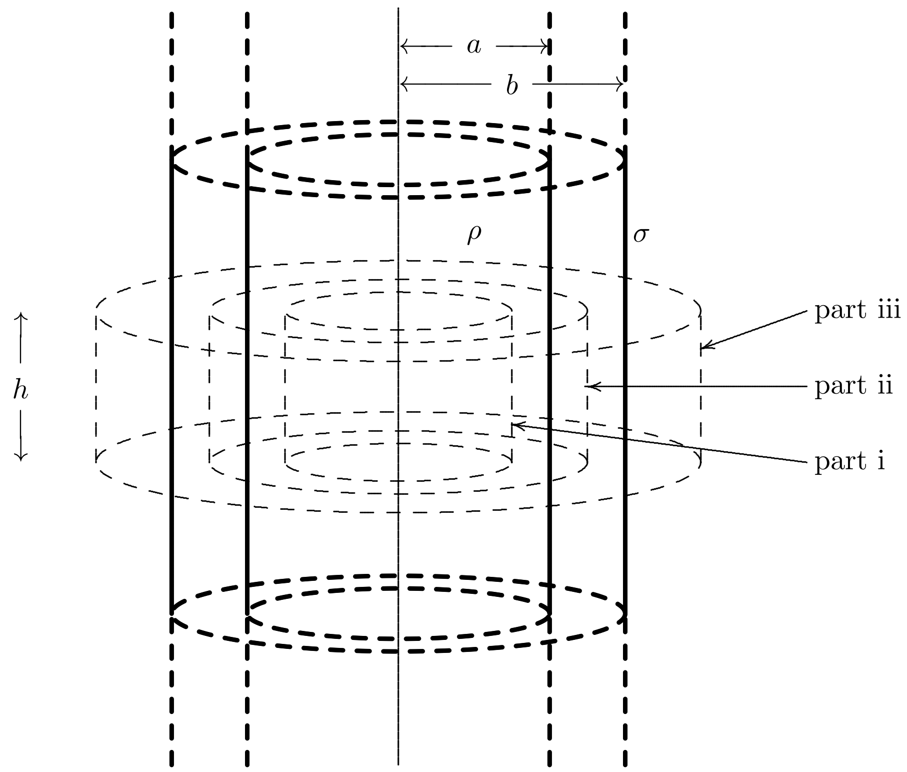
to the electric field as seen in problem 7. The shells of radii less than z produce electric fields as if their charges were concentrated at the center. So the total charge of these shells placed at the center will produce the electric field at a distance z from the center. The total charge of these shells is $q' = \rho(4\pi z^3/3)$ as their total volume is $4\pi z^3/3$. So, using the result of problem 7,

$$\vec{\mathbf{E}} = \frac{q'\hat{\mathbf{z}}}{4\pi\epsilon_0 z^2} = \frac{\rho(4\pi z^3/3)\hat{\mathbf{z}}}{4\pi\epsilon_0 z^2} = \frac{\rho z\hat{\mathbf{z}}}{3\epsilon_0}$$

As the total charge on the full sphere can be written as $q = \rho(4\pi R^3/3)$, $\rho = 3q/(4\pi R^3)$. Inserting this in the above equation gives,

$$\vec{\mathbf{E}} = \frac{qz\hat{\mathbf{z}}}{4\pi\epsilon_0 R^3}, \quad \text{for } z < R.$$

Problem 16



Charge on inner cylinder of length h is

$$q_i = \rho\pi a^2 h.$$

Charge on outer cylinder of length h is

$$q_o = \sigma(2\pi b)h.$$

The net charge being zero means

$$q_i + q_o = \rho\pi a^2 h + \sigma(2\pi b)h = 0.$$

So,

$$\sigma = -\frac{\rho a^2}{2b}.$$

For the cylindrical Gaussian surfaces shown in the figure, the flux integral is as follows.

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \int_{\text{top}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} + \int_{\text{bottom}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} + \int_{\text{curve}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}}.$$

The first two integrals vanish as the directions of $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are perpendicular. The integral over the curved part gives

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \int_{\text{curve}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \int_{\text{curve}} E\, da \cos 0^\circ = E \int_{\text{curve}} da = E 2\pi s h.$$

part i

Using the surface shown in the figure for this part, the enclosed charge is

$$Q_e = \rho \pi s^2 h.$$

Hence, from Gauss' law

$$E 2\pi s h = Q_e/\epsilon_0 = \rho \pi s^2 h/\epsilon_0,$$

and then,

$$E = \frac{\rho s}{2\epsilon_0}.$$

part ii

Using the surface shown in the figure for this part, the enclosed charge is

$$Q_e = \rho \pi a^2 h.$$

Hence, from Gauss' law

$$E 2\pi s h = Q_e/\epsilon_0 = \rho \pi a^2 h/\epsilon_0,$$

and then,

$$E = \frac{\rho a^2}{2\epsilon_0 s}.$$

part iii

Using the surface shown in the figure for this part, the enclosed charge is

$$Q_e = \rho \pi a^2 h + \sigma(2\pi b) h = \rho \pi a^2 h + \left(-\frac{\rho a^2}{2b}\right)(2\pi b) h = 0.$$

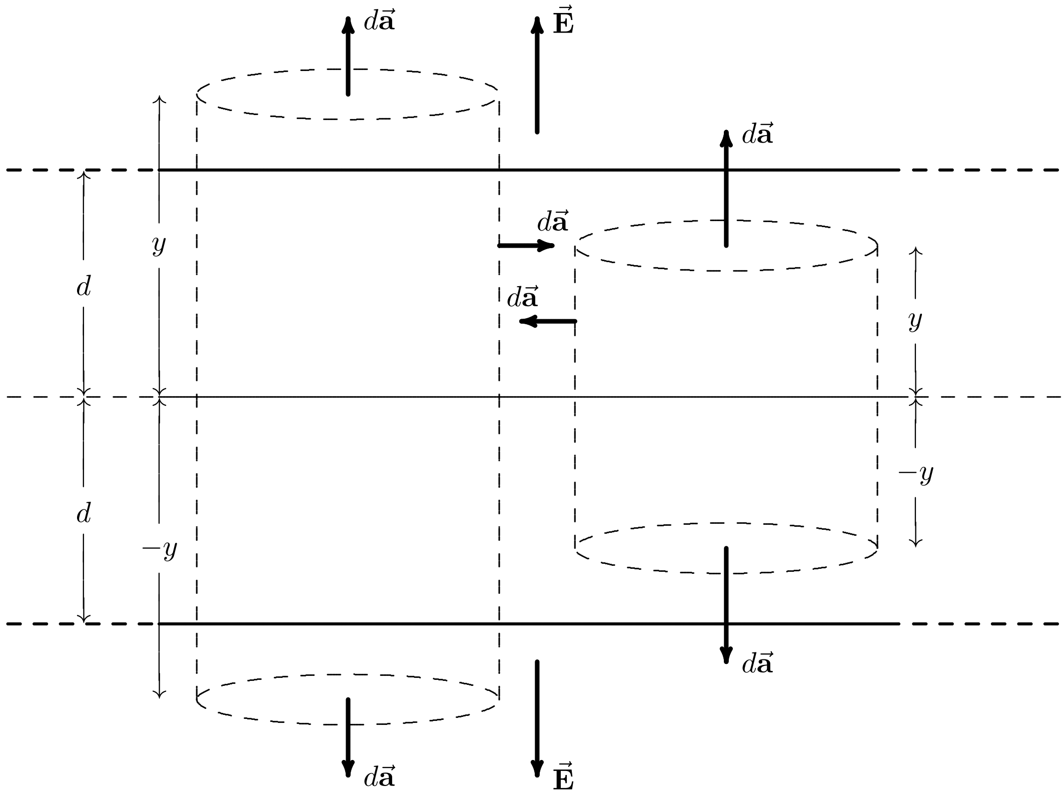
Hence, from Gauss' law

$$E 2\pi s h = Q_e/\epsilon_0 = 0,$$

and then,

$$E = 0.$$

Problem 17



The figure above shows a cross section of the charge distribution. From symmetry, it is seen that the electric field must be perpendicular to the midway plane – upwards above it and downwards below it. A convenient choice of Gaussian surface is a cylinder as shown. The electric field $\vec{\mathbf{E}}$ is perpendicular to the direction of $d\vec{\mathbf{a}}$ on the curved surface. Hence,

$$\int_{\text{curve}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = 0.$$

On the top and bottom surfaces, the angle between $\vec{\mathbf{E}}$ and $d\vec{\mathbf{a}}$ is zero. Hence,

$$\int_{\text{top}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \int_{\text{bottom}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = E \int da = Ea,$$

where a is the area of cross-section of the cylinder. This gives,

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \int_{\text{top}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} + \int_{\text{bottom}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} + \int_{\text{curve}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = 2Ea.$$

The cylinder on the left is for points such that $|y| > d$. The enclosed charge is the charge in the full thickness of the slab. So,

$$Q_{enc} = \rho a(2d).$$

So, from Gauss' law,

$$2Ea = \rho a(2d)/\epsilon_0.$$

Then,

$$E = \rho d/\epsilon_0.$$

The cylinder on the right is for points such that $|y| < d$. The enclosed charge is the charge in a thickness of $2|y|$. So,

$$Q_{enc} = \rho a(2|y|).$$

So, from Gauss' law,

$$2Ea = \rho a(2|y|)/\epsilon_0.$$

Then,

$$E = \rho|y|/\epsilon_0.$$

Problem 21

The electric field is given by,

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{qr\hat{\mathbf{r}}}{R^3} \text{ if } r < R,$$
$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{q\hat{\mathbf{r}}}{r^2} \text{ if } r > R.$$

For points outside, the electric field is given by the $r > R$ condition all along the integration path. So,

$$V(r) = -\int_{\mathcal{O}}^{\vec{\mathbf{r}}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r} \frac{q\hat{\mathbf{r}}}{r'^2} \cdot (dr'\hat{\mathbf{r}}) = -\frac{q}{4\pi\epsilon_0}\int_{\infty}^{r} \frac{1}{r'^2}dr' = \frac{q}{4\pi\epsilon_0 r}.$$

For points inside, the electric field is given by the $r > R$ condition from ∞ to R and by the $r < R$ condition from R to r along the integration path. So, the integral must be split into two parts as follows.

$$\begin{aligned} V(r) = -\int_{\mathcal{O}}^{\vec{\mathbf{r}}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} &= -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{R} \frac{q\hat{\mathbf{r}}}{r'^2} \cdot (dr'\hat{\mathbf{r}}) - \frac{1}{4\pi\epsilon_0}\int_{R}^{r} \frac{qr'\hat{\mathbf{r}}}{R^3} \cdot (dr'\hat{\mathbf{r}}) \\ = -\frac{q}{4\pi\epsilon_0}\int_{\infty}^{R} \frac{1}{r'^2}dr' - \frac{q}{4\pi\epsilon_0 R^3}\int_{R}^{r} r'dr' &= \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_0 R^3}(r^2/2 - R^2/2) \\ &= \frac{3q}{8\pi\epsilon_0 R} - \frac{qr^2}{8\pi\epsilon_0 R^3} \end{aligned}$$

Problem 22

For an infinitely long line charge with line charge density λ, the electric field is

$$\vec{\mathbf{E}} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}.$$

Choosing s_0 as a reference point and doing the integral for V along a radial line gives

$$V(s) = -\int_{\mathcal{O}}^{\vec{\mathbf{r}}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{\lambda}{2\pi\epsilon_0}\int_{s_0}^{s} \frac{\hat{\mathbf{s}}}{s'} \cdot (ds'\hat{\mathbf{s}}) = -\frac{\lambda}{2\pi\epsilon_0}\int_{s_0}^{s} \frac{ds'}{s'} = \frac{\lambda}{2\pi\epsilon_0}(\ln s_0 - \ln s).$$

Note that s_0 can be any constant as long as it is not ∞.

Problem 25

Part a

$$V = \frac{q}{4\pi\epsilon_0 r_1} + \frac{q}{4\pi\epsilon_0 r_2},$$

where

$$r_1 = r_2 = \sqrt{z^2 + (d/2)^2}$$

Hence,

$$V = \frac{2q}{4\pi\epsilon_0\sqrt{z^2 + (d/2)^2}}$$

As the x and y coordinates have been chosen as zero at the target point, derivatives with respect to them cannot be computed. Hence, only the z component of the electric field can be computed form the above expression for V.

$$E_z = -\frac{\partial V}{\partial z} = \frac{2qz}{4\pi\epsilon_0(z^2 + (d/2)^2)^{3/2}}.$$

If one of the two charges is replaced by a $-q$ charge, it is easily seen that

$$V = 0.$$

This gives the z component of the electric field to be zero. As argued before, the x and y components cannot be computed from this expression for V as it is assumed that $x = 0$ and $y = 0$.

Part b

$$V = \frac{1}{4\pi\epsilon_0}\int_{-L}^{L} \frac{\lambda dx}{(x^2 + z^2)^{1/2}} = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\sqrt{z^2 + L^2} + L}{\sqrt{z^2 + L^2} - L}\right).$$

The z component of the electric field is

$$E_z = -\frac{\partial V}{\partial z} = \frac{2\lambda L}{4\pi\epsilon_0 z\sqrt{z^2 + L^2}}$$

Part c

$$V = \frac{\sigma}{4\pi\epsilon_0}\int \frac{da}{\sqrt{x^2+y^2+z^2}}.$$

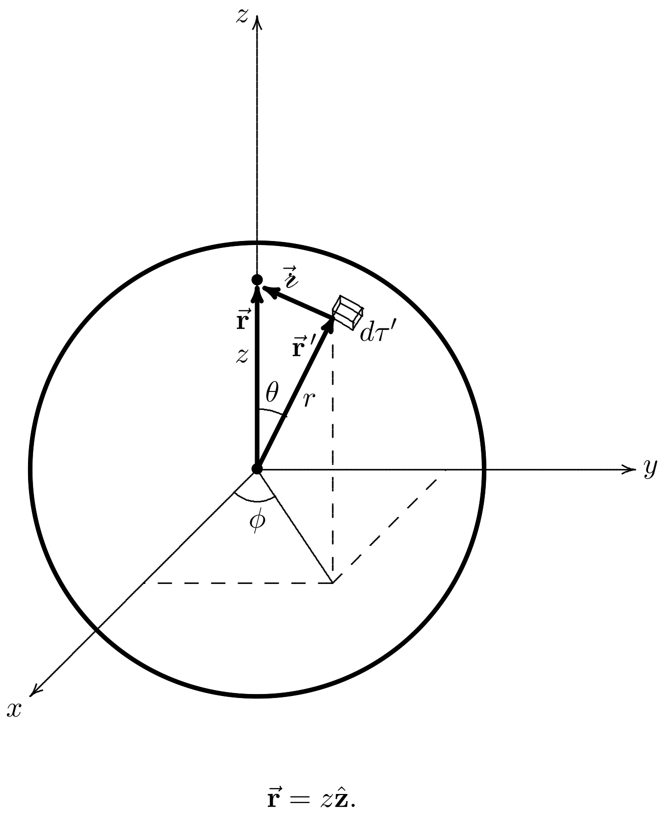
In cylindrical polar coordinates (ρ, ϕ, z), $x = \rho\cos\phi, y = \rho\sin\phi$ and $da = \rho\, d\rho\, d\phi$. Hence,

$$V = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{\rho\, d\rho\, d\phi}{\sqrt{\rho^2+z^2}} = \frac{2\pi\sigma}{4\pi\epsilon_0}\int_0^R \frac{\rho\, d\rho}{\sqrt{\rho^2+z^2}} = \frac{\sigma}{2\epsilon_0}(\sqrt{R^2+z^2}-z).$$

The z component of the electric field is

$$E_z = -\frac{\partial V}{\partial z} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right).$$

Problem 28



$$\vec{\mathbf{r}} = z\hat{\mathbf{z}}.$$

Converting rectangular to spherical polar coordinates for the volume element shown,

$$\vec{\mathbf{r}}\,' = r\sin\theta\cos\phi\hat{\mathbf{x}} + r\sin\theta\sin\phi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}},$$

$$\vec{\boldsymbol{\mathscr{r}}} = \vec{\mathbf{r}} - \vec{\mathbf{r}}\,' = -r\sin\theta\cos\phi\hat{\mathbf{x}} - r\sin\theta\sin\phi\hat{\mathbf{y}} + (z - r\cos\theta)\hat{\mathbf{z}}.$$

Hence,

$$\gamma = (r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + z^2 + r^2 \cos^2\theta - 2rz\cos\theta)^{1/2} = (r^2 + z^2 - 2rz\cos\theta)^{1/2},$$

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\, d\tau'}{\gamma} = \frac{\rho}{4\pi\epsilon_0} \int_0^R \int_0^{2\pi} \int_0^{\pi} \frac{r^2 \sin\theta\, d\theta\, d\phi\, dr}{(r^2 + z^2 - 2rz\cos\theta)^{1/2}}.$$

The ϕ integral gives

$$\int_0^{2\pi} d\phi = 2\pi.$$

Using the substitution $u = (r^2 + z^2 - 2rz\cos\theta)^{1/2}$, the θ integral gives

$$\int_0^{\pi} \frac{\sin\theta\, d\theta}{(r^2 + z^2 - 2rz\cos\theta)^{1/2}} = \frac{1}{rz}\left[\sqrt{(r+z)^2} - \sqrt{(r-z)^2}\right]$$

$$= \begin{cases} 2/r & \text{if } r > z, \\ 2/z & \text{if } r < z. \end{cases}$$

Hence,

$$V = \frac{4\pi\rho}{4\pi\epsilon_0}\left[\int_0^z \frac{r^2 dr}{z} + \int_z^R \frac{r^2 dr}{r}\right] = \frac{\rho}{\epsilon_0}[z^2/3 + R^2/2 - z^2/2] = \frac{\rho}{\epsilon_0}[R^2/2 - z^2/6]$$

As the total charge is $q = \rho(4\pi R^3/3)$,

$$V = \frac{3q}{4\pi\epsilon_0 R^3}[R^2/2 - z^2/6] = \frac{3q}{8\pi\epsilon_0 R} - \frac{qz^2}{8\pi\epsilon_0 R^3}.$$

Problem 34

Part a

As $\rho = 0$ outside the sphere, the integral needs to be done only inside the sphere. The potential V inside the sphere is found to be (problem 21),

$$V = \frac{3q}{8\pi\epsilon_0 R} - \frac{qr^2}{8\pi\epsilon_0 R^3}, \quad \text{if } r < R.$$

So,

$$W = \frac{\rho}{2}\int_0^R \int_0^{2\pi} \int_0^{\pi} \left[\frac{3q}{8\pi\epsilon_0 R} - \frac{qr^2}{8\pi\epsilon_0 R^3}\right] r^2 \sin\theta\, d\theta\, d\phi\, dr,$$

$$= 2\pi\rho \int_0^R \left[\frac{3q}{8\pi\epsilon_0 R} - \frac{qr^2}{8\pi\epsilon_0 R^3}\right] r^2 dr = \rho\frac{qR^2}{5\epsilon_0} = \frac{3q^2}{20\pi\epsilon_0 R},$$

where the fact that $q = \rho(4\pi R^3/3)$ is used.

Part b

$$E = \begin{cases} \frac{qr}{4\pi\epsilon_0 R^3} & \text{if } r < R, \\ \frac{q}{4\pi\epsilon_0 r^2} & \text{if } r > R. \end{cases}$$

Hence,

$$W = \frac{\epsilon_0}{2}\int_0^\infty \int_0^{2\pi} \int_0^\pi E^2 r^2 \sin\theta \, d\theta \, d\phi \, dr = 2\pi\epsilon_0 \int_0^\infty E^2 r^2 \, dr = 2\pi\epsilon_0 \left[\int_0^R E^2 r^2 \, dr + \int_R^\infty E^2 r^2 \, dr\right]$$

$$= 2\pi\epsilon_0 \left[\int_0^R \left(\frac{qr}{4\pi\epsilon_0 R^3}\right)^2 r^2 \, dr + \int_R^\infty \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 r^2 \, dr\right] = \frac{q^2}{8\pi\epsilon_0}\left[\frac{1}{5R} + \frac{1}{R}\right] = \frac{3q^2}{20\pi\epsilon_0 R}$$

Part c

Let $a > R$. Then, the volume integral part is the same as above except for the upper limit of integration being a rather than ∞. So,

$$\int_{\mathcal{V}} E^2 \, d\tau = 4\pi \left[\int_0^R \left(\frac{qr}{4\pi\epsilon_0 R^3}\right)^2 r^2 \, dr + \int_R^a \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 r^2 \, dr\right] = \frac{q^2}{4\pi\epsilon_0^2}\left[\frac{1}{5R} + \frac{1}{R} - \frac{1}{a}\right]$$

The surface is a sphere of radius a (do not confuse a with the vector $d\vec{\mathbf{a}}$ which is a surface area element). $d\vec{\mathbf{a}}$ is parallel to $\vec{\mathbf{E}}$ and hence,

$$\vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = E|d\vec{\mathbf{a}}| = Ea^2 \sin\theta \, d\theta \, d\phi$$

So, the surface integral is

$$\oint_{\mathcal{S}} V\vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \int_0^{2\pi} \int_0^\pi VEa^2 \sin\theta \, d\theta \, d\phi,$$

As V and E depend only on a, the radial distance,

$$\oint_{\mathcal{S}} V\vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = VEa^2 \int_0^{2\pi} \int_0^\pi \sin\theta \, d\theta \, d\phi = VEa^2(4\pi) = 4\pi a^2 \frac{q}{4\pi\epsilon_0 a} \frac{q}{4\pi\epsilon_0 a^2} = \frac{q^2}{4\pi\epsilon_0^2 a}$$

Then it can be seen that

$$W = \frac{\epsilon_0}{2}\left(\int_{\mathcal{V}} E^2 \, d\tau + \oint_{\mathcal{S}} V\vec{\mathbf{E}} \cdot d\vec{\mathbf{a}}\right) = \frac{\epsilon_0}{2}\left(\frac{q^2}{4\pi\epsilon_0^2}\left[\frac{1}{5R} + \frac{1}{R} - \frac{1}{a}\right] + \frac{q^2}{4\pi\epsilon_0^2 a}\right) = \frac{3q^2}{20\pi\epsilon_0 R}$$

Problem 35

At the stage where the sphere has been built to a radius of r, the potential at the surface is

$$V(r) = \frac{q'}{4\pi\epsilon_0 r}$$

where q' is the total charge of the sphere at that stage, that is

$$q' = \rho(4\pi r^3/3)$$

Hence,

$$V(r) = \frac{\rho(4\pi r^3/3)}{4\pi\epsilon_0 r} = \frac{\rho r^2}{3\epsilon_0}$$

The thin shell of thickness dr will have a charge of

$$dq' = \rho\, d\tau = \rho(4\pi r^2 dr)$$

The work done to bring that charge to the surface will be

$$dW = V\, dq' = \frac{\rho r^2}{3\epsilon_0}\rho(4\pi r^2 dr) = \frac{4\pi\rho^2}{3\epsilon_0} r^4 dr$$

Integrating this from 0 to R gives the total energy.

$$W = \int dW = \int_0^R \frac{4\pi\rho^2}{3\epsilon_0} r^4 dr = \frac{4\pi\rho^2 R^5}{15\epsilon_0}$$

Then using the fact that $q = \rho(4\pi R^3/3)$, we get

$$W = \frac{3q^2}{20\pi\epsilon_0 R}.$$