# Fault-Tolerant Design of Digital Systems

## EGE 534

### Evaluation Techniques

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## Outline

- Evaluation Techniques
  - Reliability
  - Mean Time to Failure
- Combinatorial Model
  - Series
  - Parallel
  - M-of-N
  - Non-Series and Non-Parallel
- Markov Model
  - Discrete time
  - Continuous time

## Evaluation

- Allows comparison of design techniques and subsequent tradeoffs
  - Qualitative
    - Subjective – the benefit of one system over the other
  - Quantitative
    - Produces numbers to compare systems
- Quantitative Models: vital means for system reliability and availability predictions
  - Measure failure rate
  - Calculate reliability, availability, MTTF …
- Importance of analysis and analytical model
  - Evaluate a design and compare different designs
  - Feedback to the designer during early design stages
  - Use the model for performance analysis

## Reliability Function

- Earlier definitions
  - Reliability R(t)
  - Availability A(t)
- Failure rate
  - The expected number of failures of a type of device per given time
  - Device fails once every 2000 hours, $\lambda$ (failure rate) = 1/2000 failures/hour
- Consider a large experiment with N systems at time $t_0$
  - $N_0(t)$ - number of correctly operating systems at time $t$
  - $N_f(t)$ - number of failed systems at time $t$
  - Reliability

$$R(t) = \frac{N_o(t)}{N} = \frac{N_o(t)}{N_o(t) + N_f(t)}$$

$$R(t) = 1 - \frac{N_f(t)}{N}$$

  - Unreliability Q(t) = 1 - R(t)

$$Q(t) = \frac{N_f(t)}{N} = \frac{N_f(t)}{N_o(t) + N_f(t)}$$

## Reliability Function

- Derivative of reliability

$$\frac{d\,\mathrm{R(t)}}{dt} = -\frac{1}{N}\frac{d\mathrm{N_f(t)}}{d\mathrm{t}}$$

- $\frac{d\mathrm{N_f(t)}}{d\mathrm{t}}$ is called instantaneous failure rate of the component

- Divide by $\mathrm{N_o(t)}$

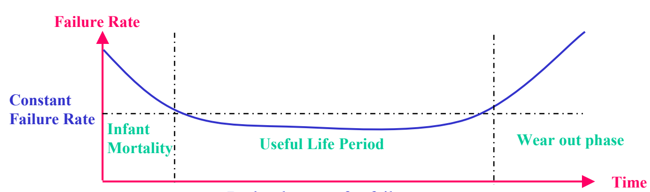
$$z(t) = \frac{1}{\mathrm{N_o(t)}}\frac{d\mathrm{N_f(t)}}{dt}$$

  - Z(t) is called hazard or hazard rate, or failure rate

$$z(t) = \frac{1}{\mathrm{N_O(t)}}\left(-N\frac{d\mathrm{R(t)}}{dt}\right) = -\frac{1}{R(t)}\frac{d\mathrm{R(t)}}{dt}$$

$$z(t) = -\frac{1}{\mathrm{R(t)}}\frac{dR\mathrm{(t)}}{dt} \rightarrow \frac{dR\mathrm{(t)}}{dt} = -z(t)\mathrm{R(t)}$$

## Reliability Function



Bath tub curve for failure rate

- Constant failure rate during useful life
- Infant mortality and wear out periods have variable failure rate
- Reliability computation - constant failure rate

$$z(t) = \lambda \rightarrow \frac{dR(t)}{dt} = -\lambda R(t)$$

$$R(t) = e^{-\lambda t}$$

Time varying failure rate

Waibull distribution $z(t) = \alpha\lambda(\lambda t)^{(\alpha-1)}$

$\alpha = 1 \rightarrow z = \lambda$

$\alpha > 1 \rightarrow$ z increases with time

$\alpha < 1 \rightarrow$ z decreases with time

$\mathrm{R(t)} = e^{-(\lambda t)^{\alpha}}$

## Mean Time to Failure (MTTF)

- MTTF: *expected time* that a system will operate before the first failure occurs
- Assume N identical systems operational at $t_0$

$$\text{MTTF} = \frac{1}{\text{N}}\sum_{i=1}^{N} t_i$$

  - $t_i$ is the measure time that item *i* operates before failing
- Probability theory
  - Expected value of a random variable $x$ is

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

**f(x) Probability density function**

  - MTTF = expected value of time failure = $\int_{-\infty}^{+\infty} t f(t) dt = \int_{0}^{+\infty} t f(t) dt$

$$f(t) = \frac{dQ(t)}{dt}$$

**f(t) failure density function**

## Mean Time to Failure (MTTF)

$$MTTF = \int_0^{\infty} t \frac{dQ(t)}{dt} dt$$

- Integration by parts and the fact that $\frac{dQ(t)}{dt} = -\frac{dR(t)}{dt}$

$$MTTF = \int_0^{\infty} -t \frac{dR(t)}{dt} dt = \left[ -tR(t) + \int_0^{\infty} R(t) dt \right] \Bigg|_0^{\infty}$$

=0 at t=0
=0 at t = ∞ → R(t) =0

$$MTTF = \int_0^{\infty} R(t) dt$$

Valid for any system with R(∞) =0

## **Mean Time to Failure** (MTTF)

- Example: MTTF for a system with exponential failure law

$$MTTF = \int_0^{\infty} e^{-\lambda t} dt = -\frac{1}{\lambda} e^{-\lambda t} \Big|_0^{\infty}$$

$$MTTF = 0 - (-\frac{1}{\lambda}) = \frac{1}{\lambda}$$

- Reliability at t = MTTF

$$R(MTTF) = R(\frac{1}{\lambda}) = e^{-\lambda(\frac{1}{\lambda})} = e^{-1} = 0.3678$$

## MTTR and MTBF

- Mean time to repair – MTTR
  - Assume constant repair rate ($\mu$) and arguments similar to those used for failure analysis and conclude MTTR = $1/\mu$
- Mean time between failure – MTBF

$$MTBF = \frac{\text{total time T}}{\text{average number of failures}}$$

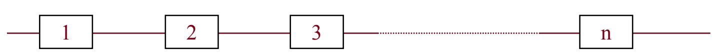
  - Use heuristic arguments to conclude can also argue MTBF = MTTF + MTTR
  - Note: often $\lambda \ll \mu$ and hence MTTF >> MTTR , therefore the words MTTF and MTBF are used improperly by some

# Combinatorial Modeling

- System is divided into non-overlapping modules
- Each module is assigned either a probability of working, $P_i$, or a probability as function of time, $R_i(t)$
- The goal is to derive the probability, $P_{sys}$, or function $R_{sys}(t)$: Prob that the system survives until time t
- Assumptions:
  - Module failures are independent
  - Once a module has failed, it is always assumed to yield incorrect results
  - System considered failed if it does not contain a minimal set of functioning modules
  - Once system enters a failed state, other failures cannot return system to functional state
- Models typically enumerate all the states of the system that meet or exceed the requirements for a correctly functioning system
- Combinatorial counting techniques are used to simplify this process

# Series Systems

- Assume system has n components, e.g. CPU, memory, disk, terminal
- All components should survive for the system to operate correctly




$$R_{sys} = R_1 \bullet R_2 \bullet R_3 \bullet \ldots \bullet R_n$$

- Reliability of the system

$$R_{series}(t) = \prod_{i=1}^{n} R_i(t)$$ where $R_i(t)$ is the reliability of module i

- For exponential failure rate of each component

$$R_{series}(t) = e^{-\sum_{i=1}^{n} \lambda_i t} = e^{-\lambda_{system} t}$$

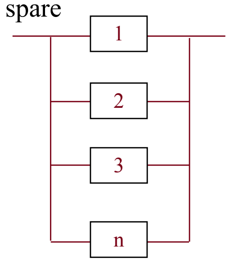
where $\lambda_{system} = \sum_{i=1}^{n} \lambda_i$ corresponds to the failure rate of the system

Effect is summation of failure rates of components

## Parallel Systems

- Assume system with spares
- As soon as fault occurs a faulty component is replaced by a spare
- Only one component needs to survive for the system to operate correctly
- Prob. of module $i$ to survive = $R_i$
- Prob., module $i$ does not survive = $(1 - R_i)$
- Prob. of no modules survive = $(1 - R_1)(1 - R_2)_{\dots}(1 - R_n)$

  Prob [at least one module survives] = 1 - Prob [no module survives]



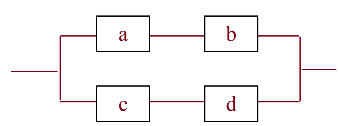

- Reliability of the parallel system

$$R_{parallel}(t) = 1.0 - \prod_{i=1}^{n}\left(1.0 - R_i(t)\right)$$

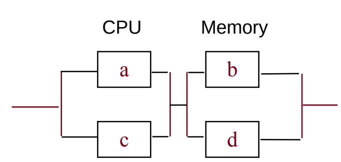
## Series-Parallel Systems

- Consider combinations of series and parallel systems
- Example, two CPUs connected to two memories in different ways

$R_{sys} = 1 - (1-Ra\ Rb)\ (1-Rc\ Rd)$

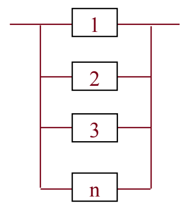



$R_{sys} = (1-(1-Ra)(1-Rc))\ (1-(1-Rb)(1- Rd))$



## A Simple Example

- Consider dynamic redundant system with spares (dynamic redundancy)
- As soon as fault occurs, a faulty component is replaced by a spare
- Up to (n-1) spare modules
- $R_{sys} = 1 - (1-R_1)(1-R_2)\ldots(1-R_n)$
- Example: Consider identical modules with $R_i = 0.9$
- How can you increase $R_{sys}$ to $0.999999 = 1-10^{-6}$



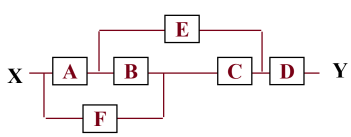

$$1-10^{-6} = 1-(1-0.9)^n$$

$$n = \frac{\ln(10^{-6})}{\ln(1-0.9)} = 6$$

- Hence, need 5 spares to make reliable system

## Non-Series-Parallel-Systems

- "Success" diagram used to represent the operational modes of the system



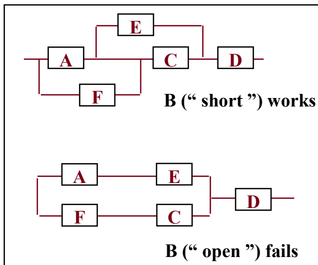

Each path from X to Y represents a configuration that leaves the system successfully operational

- Reliability of the system derived by expanding around a single module *m*

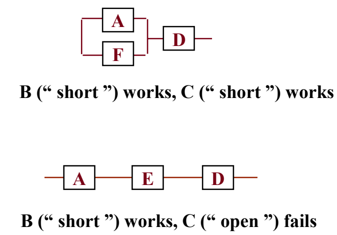
$R_{sys} = R_m$ P (system works|$m$ works) + $(1-R_m)$ P (system works|$m$ fails)

P(s|m) denotes the conditional probability "s given, *m* has occurred"

## Non-Series-Parallel-Systems (cont.)




***Reduced model with B replaced***




***Reduction with B and C replaced***

$R_{sys} = R_B\, P(\text{system works}|\text{B works})$
$+ (1 - R_B)\ \{R_D[1 - (1 - R_A R_E)(1 - R_F R_C)]\}$

$P(\text{system works}|\text{B works}) =$
$R_C\{R_D[1 - (1 - R_A)(1 - R_F)]\}$
$+ (1 - R_C)(R_A R_D R_E)$

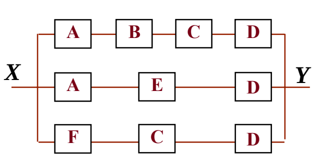
**Letting $R_A$ ....$R_F$ = $R_m$ yields** $\quad$ $R_{sys} = R^6_m - 3R^5_m + R^4_m + 2R^3_m$

## Non-Series-Parallel-Systems (cont.)

- For complex success diagrams, an upper-limit approximation on $R_{sys}$ can be used
- An upper bound on system reliability is:

$$R_{sys} \le 1 - \prod \left(1 - R_{path\ \ i}\right)$$

$R_{path}$ is the serial reliability of path i




***Reliability block diagram (RBD) of a system***

$R_{sys}$ equation is an upper bound because the paths are not independent.
That is, the failure of a single module affects more than one path.

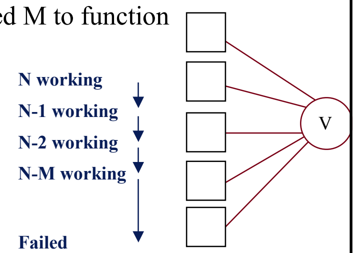
$$R_{sys} \le 1 - (1 - R_A R_B R_C R_D)(1 - R_A R_E R_D)(1 - R_F R_C R_D)$$
$$R_{sys} \le 2R^3_m + R^4_m - R^6_m - 2R^7_m + R^{10}_m$$

## M-out-of-N Systems

- Static or masking redundancy
- General *M-out-of-N* system, out of N modules, need M to function
  - System can tolerate N-M failures

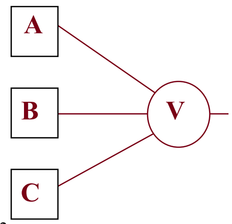
$$R_{MN} = \sum_{i=0}^{N-M} \binom{N}{i} R_m^{N-i} Q_m^{\ i}$$

  - Where $\binom{N}{i} = \frac{N!}{(N-i)!i!}$




- Example: Consider *TMR* (Triple Modular Redundancy) system *(2-out-of-3)*

$$R_{MN} = \sum_{i=0}^{1} \binom{3}{i} R_m^{3-i} Q_m^{\ i}$$




$$R_{TMR}(t) = R_m^3(t) + \binom{3}{1} R_m^2(t)(1 - R_m(t)) = 3R_m^2(t) - 2R_m^3(t)$$

## Reliability of TMR VS. Singular

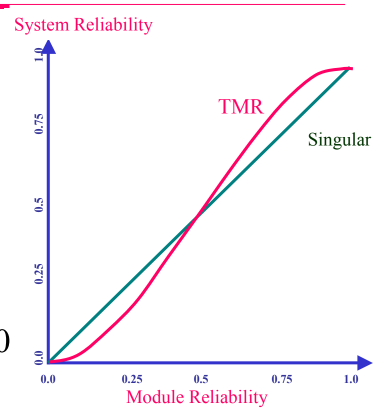
$$R_{TMR}(t) = 3R^2(t) - 2R^3(t)$$

Find the cross points

$$R = 3R^2 - 2R^3$$

$$3R^2 - 2R^3 - R = 0$$

$$(R - .5)(R - 1) = 0 \rightarrow R = 0.5, R = 1.0$$

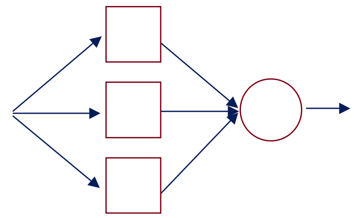



- A system can tolerate fault and still have low reliability.
- A system can achieve a high reliability without being fault tolerant

## Effect of Voter

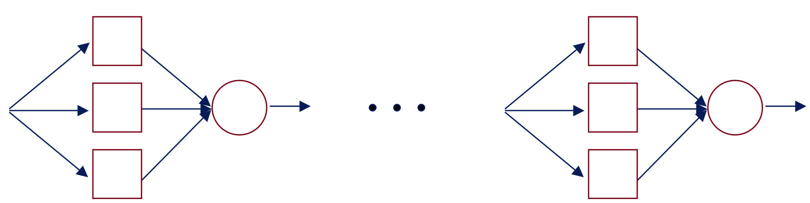
- ☐ Previous expression for reliability assumed voter 100% reliable
- ☐ Assume voter reliability $R_v$

$$R_{TMRV} = R_V(3R_m^2 - 2R_m^3)$$



## Cascading TMR Systems

- Consider n stages of original system
- Each stage replaced by TMR with Voter

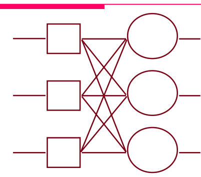


Reliability of the system

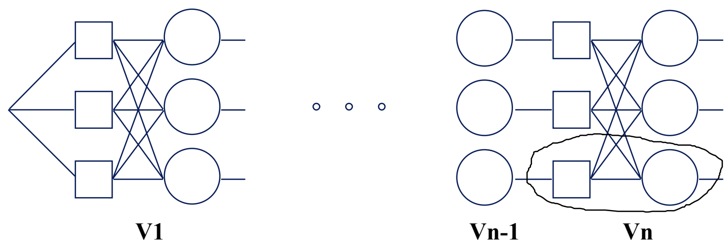
$$R_{cascade} = \left(R_V\left(3R_m^2 - 2R_m^3\right)\right)^n$$

# TMR with 3 Voters

- Remove single point of failure
- Use TMR with 3 voters
- Cascade such systems



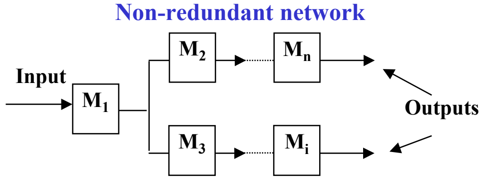
$$R_{TMRV} = 3(R_V R_m)^2 - 2(R_V R_m)^3$$



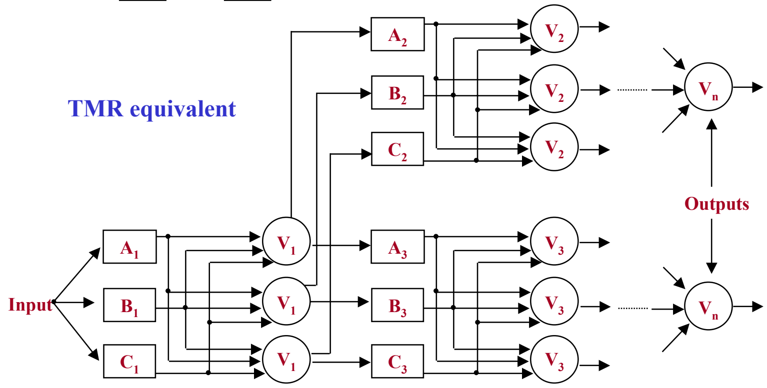

Reliability expression can be obtained by considering *module -voter* combination as a unit; then apply TMR expression. $R_{TMRV} = (3(R_V R_m)^2 - 2(R_V R_m)^3)^n$

# TMR in Complex Networks

**Non-redundant network**




**TMR equivalent**



## Pitfalls Using Single Model

- Should we use MTTF to compare the reliability of systems
- Compare reliability of simplex and TMR systems (assume perfect voter)

$R_{simplex}(t) = e^{-\lambda t}$

$$MTTF_{simplex} = \int_0^{\infty} e^{-\lambda t} dt = \frac{1}{\lambda}$$

$$R_{TMR}(t) = R^3(t) + \binom{3}{1} R^2(t)(1 - R(t)) = 3R^2(t) - 2R^3(t)$$

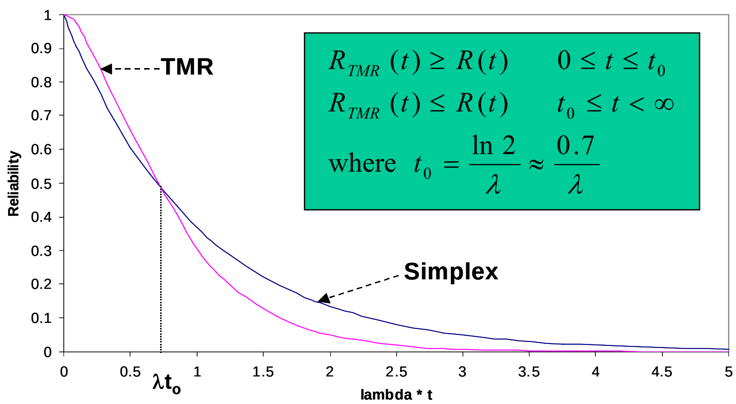
$$R_{TMR}(t) = 3e^{-2\lambda t} - 2e^{-3\lambda t}$$

$$MTTF_{TMR}(t) = \int_0^{\infty} (3e^{-2\lambda t} - 2e^{-3\lambda t}) dt = -\frac{3}{2\lambda} e^{-2\lambda t} + \frac{2}{3\lambda} e^{-3\lambda t} \Big|_{\infty}^{0}$$

$$MTTF_{TMR} = \frac{3}{2\lambda} - \frac{2}{3\lambda} = \frac{5}{6\lambda}$$

$MTTF_{simplex} > MTTF_{TMR}$ ➔ May conclude that TMR is not as good as simplex

## Pitfalls Using Single Model (cont.)




$$R_{TMR}(t) \geq R(t) \qquad 0 \leq t \leq t_0$$
$$R_{TMR}(t) \leq R(t) \qquad t_0 \leq t < \infty$$
$$\text{where } t_0 = \frac{\ln 2}{\lambda} \approx \frac{0.7}{\lambda}$$

- MTTF is the area under the reliability curve
- There is a point that TMR becomes less reliable than simplex
- MTTF can sometimes misrepresent the quality of a system
- Regardless, if fault-tolerance is necessary, TMR will be superior

## Pitfalls Using Single Model (cont.)

- Instead of MTTF, look at mission time
- Reliability of M-out-of-N systems very high in the beginning
  - Spare components tolerate failures
- Reliability sharply falls down in end
  - System exhausted redundancy, more hardware can possibly fail
- Such systems useful in aircraft control
  - Very high reliability, short time
  - 0.99999 over 10 hour period
- Improving "Vanilla" TMR: TMR with Recovery (e.g. Tandem), TMR Simplex (Run in usual TMR mode.. On first failure drop the second component and run in Simplex)

## Effect of Coverage

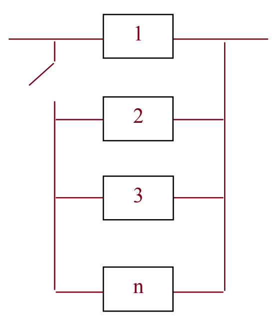
- Failure detection is not perfect
- Reconfiguration may not succeed
- Attach a coverage "c"

One spare system

$R_{sys} = R_1 + c\,(1-R_1)\,R_2$

n-1 spare system

$$R_{sys} = R_m \sum_{i=0}^{n-1} c^i \left(1 - R_m\right)^i$$



## Effect of Coverage (cont.)

- If coverage is 100%, then given low module reliability, can increase system reliability arbitrarily

With low coverage, reliability saturates

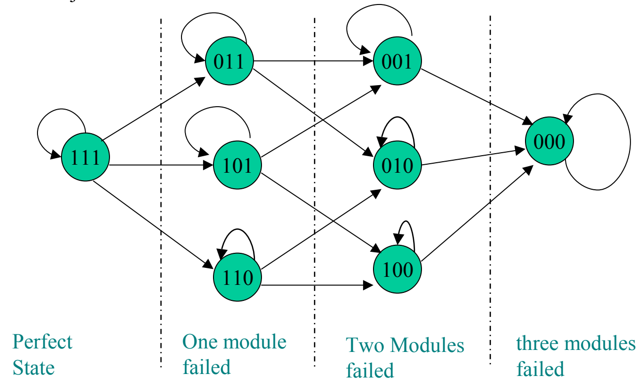
| | Rm = 0.9 | Rm = 0.7 | Rm = 0.5 |
|---|---|---|---|
| C=0.99, n=2 | 0.989 | 0.908 | 0.748 |
| C=0.99, n=4 | 0.999 | 0.988 | 0.931 |
| C=0.99, n=inf | 0.999 | 0.996 | 0.990 |
| | | | |
| C= 0.8 , n=2 | 0.972 | 0.868 | 0.700 |
| C= 0.8 , n=4 | 0.978 | 0.918 | 0.812 |
| C=0.8, n=inf | 0.978 | 0.921 | 0.833 |

## Markov Models

- Difficulty with combinatorial models
  - Many complex systems cannot be modeled easily
    - Reliability block diagram difficult to construct
  - Incorporating repairs in the model and analysis
  - Incorporation of coverage factor – such as in duplicates system we may be less than 100% certain that only faulty unit will be eliminated when system is re-configured
- Main concept of Markov Model
  - System State
  - State transition

## Markov Model (Cont.)

- Example: the concept of state using TMR (8 states)
  - Transitions between states occur with certain probabilities
  - Markov model assumption: Probability of transition from a state $s_i$ to $s_j$ is independent of the method of arrival into state $s_i$




## Markov Model- Reduced

- Assumption: each module obeys exponential failure law with λ failures
- Probability of a module being failed at some point t+ Δ t, given that it was operational at time t is Q(Δ t) = 1 – R(Δ t) = 1 – $e^{-\lambda (\Delta t)}$

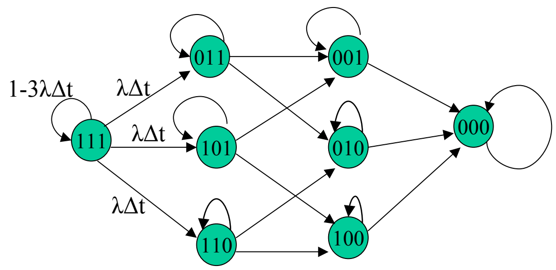
$$e^{-\lambda \Delta t} = 1 + (-\lambda \Delta t) + \frac{(-\lambda \Delta t)^2}{2!} + \ldots$$

$$1 - e^{-\lambda \Delta t} = (\lambda \Delta t) - \frac{(-\lambda \Delta t)^2}{2!} + ..$$

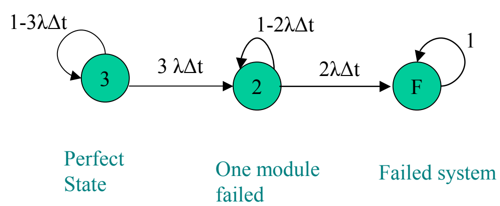
- At small Δ t

$$1 - e^{-\lambda \Delta t} = \lambda \Delta t$$

## Markov Model- Reduced (Cont.)

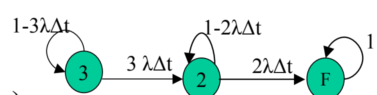



- Reduced Markov model for a TMR system by collapsing the states




## Markov Model- Transition Matrix

- Probability of the TMR being in state 3 depends on probability that it was in state 3 at *t* and the probability that it will transition back to state 3.




$$P_3(t+\Delta t) = (1-3\lambda\Delta t)P_3(t)$$
$$P_2(t+\Delta t) = 3\lambda\Delta t P_3(t) + (1-2\lambda\Delta t)P_2(t)$$
$$P_F(t+\Delta t) = 2\lambda\Delta t P_2(t) + P_F(t)$$

- Hence Transition matrix from Markov chain

$$\begin{bmatrix} P_3(t+\Delta t) \\ P_2(t+\Delta t) \\ P_F(t+\Delta t) \end{bmatrix} = \begin{bmatrix} (1-3\lambda\Delta t) & 0 & 0 \\ 3\lambda\Delta t & (1-2\lambda\Delta t) & 0 \\ 0 & 2\lambda\Delta t & 1 \end{bmatrix} \begin{bmatrix} P_3(t) \\ P_2(t) \\ P_F(t) \end{bmatrix}$$

## Markov Model- Transition Matrix

- In compact form $\mathrm{P}(t+\Delta t) = \mathrm{A}\,\mathrm{P}(t)$
- At t = 0 $\mathrm{P}(\Delta t) = \mathrm{A}\,\mathrm{P}(0)$
- $\mathrm{P}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- At t =2 Δt $\mathrm{P}(2\Delta t) = \mathrm{A}\,\mathrm{P}(\Delta t) = \mathrm{A}^2\,\mathrm{P}(0)$
- At t =n Δt $\mathrm{P}(n\Delta t) = \mathrm{A}^n\,\mathrm{P}(0)$
- $R_{TMR}(t) = P_3(t) + P_2(t)$

## Markov Model- Closed Form Solution

$$P_3(t+\Delta t) = (1-3\lambda\Delta t)P_3(t)$$
$$P_2(t+\Delta t) = 3\lambda\Delta t P_3(t) + (1-2\lambda\Delta t)P_2(t)$$
$$P_F(t+\Delta t) = 2\lambda\Delta t P_2(t) + P_F(t)$$

$$\frac{\mathrm{P}_3(t+\Delta t)-\mathrm{P}_3(t)}{\Delta t} = -3\lambda\,\mathrm{P}_3(t)$$
$$\frac{\mathrm{P}_2(t+\Delta t)-\mathrm{P}_2(t)}{\Delta t} = 3\lambda\,\mathrm{P}_3(t) - 2\lambda\,\mathrm{P}_2(t)$$
$$\frac{\mathrm{P}_F(t+\Delta t)-\mathrm{P}_F(t)}{\Delta t} = 2\lambda\,\mathrm{P}_2(t)$$

$\Delta t \to 0$

$$\frac{d\,\mathrm{P}_3(t)}{dt} = -3\lambda\,\mathrm{P}_3(t)$$
$$\frac{d\,\mathrm{P}_2(t)}{dt} = 3\lambda\,\mathrm{P}_3(t) - 2\lambda\,\mathrm{P}_2(t)$$
$$\frac{d\,\mathrm{P}_F(t)}{dt} = 2\lambda\,\mathrm{P}_2(t)$$

Solving the equations using Laplace Transform

$$S\,\mathrm{P}_3(s) - \mathrm{P}_3(0) = -3\lambda\,\mathrm{P}_3(s)$$
$$S\,\mathrm{P}_2(s) - \mathrm{P}_2(0) = 3\lambda\,\mathrm{P}_3(s) - 2\lambda\,\mathrm{P}_2(s)$$
$$S\,\mathrm{P}_F(s) - \mathrm{P}_F(0) = 2\lambda\,\mathrm{P}_2(s)$$

$$t = 0 \to \mathrm{P}_3(0) = 1; \mathrm{P}_2(0) = \mathrm{P}_F(0) = 0$$
$$\mathrm{P}_3(s) = \frac{1}{s+3\lambda} \quad \mathrm{P}_2(s) = \frac{3\lambda}{(s+2\lambda)(s+3\lambda)}$$
$$\mathrm{P}_F(s) = \frac{6\lambda^2}{s(s+2\lambda)(s+3\lambda)}$$

## Markov Model- Closed Form Solution

$$P_3(s) = \frac{1}{s + 3\lambda}$$

$$P_2(s) = \frac{3}{(s + 2\lambda)} - \frac{3}{(s + 3\lambda)}$$

$$P_F(s) = \frac{1}{s} - \frac{3}{s + 2\lambda} + \frac{2}{s + 3\lambda}$$

$$P_3(t) = e^{-3\lambda t}$$

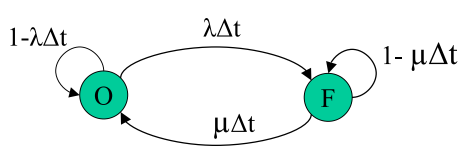
$$P_2(t) = 3e^{-2\lambda t} - 3e^{-3\lambda t}$$

$$P_F(t) = 1 - 3e^{-2\lambda t} + 2e^{-3\lambda t}$$

$$R_{TMR} = P_3(t) + P_2(t) = 3e^{-2\lambda t} - 2e^{-3\lambda t}$$

## Markov Model – Incorporating Spares

- Example: Singular system with λ failure rate and μ repair rate




- The state matrix equation

$$\begin{bmatrix} P_o(t+\Delta t) \\ P_F(t+\Delta t) \end{bmatrix} = \begin{bmatrix} (1-\lambda\Delta t) & \mu\Delta t \\ \lambda\Delta t & (1-\mu\Delta t) \end{bmatrix} \begin{bmatrix} P_0(t) \\ P_F(t) \end{bmatrix}$$

## Markov Model- Closed Form Solution

$$\begin{bmatrix} P_o(t+\Delta t) \\ P_F(t+\Delta t) \end{bmatrix} = \begin{bmatrix} (1-\lambda\Delta t) & \mu\,\Delta t \\ \lambda\Delta t & (1-\mu\,\Delta t) \end{bmatrix} \begin{bmatrix} P_0(t) \\ P_F(t) \end{bmatrix}$$

$$\frac{\mathrm{P}_O(t+\Delta t)-\mathrm{P}_O(t)}{\Delta t} = -\lambda\,\mathrm{P}_O(t)+\mu\,\mathrm{P}_F(t)$$
$$\frac{\mathrm{P}_F(t+\Delta t)-\mathrm{P}_F(t)}{\Delta t} = \lambda\,\mathrm{P}_O(t)-\mu\,\mathrm{P}_F(t)$$

$\Delta t \to 0$

$$\frac{d\,\mathrm{P}_O(t)}{dt} = -\lambda\,\mathrm{P}_O(t)+\mu\,\mathrm{P}_F(t)$$
$$\frac{d\,\mathrm{P}_F(t)}{dt} = \lambda\,\mathrm{P}_O(t)-\mu\,\mathrm{P}_F(t)$$

Solving the equations using Laplace Transform

$$S\mathrm{P}_O(s)-\mathrm{P}_O(0) = -\lambda\,\mathrm{P}_O(s)+\mu\mathrm{P}_F(s)$$
$$S\,\mathrm{P}_F(s)-\mathrm{P}_F(0) = \lambda\,\mathrm{P}_O(s)-\mu\,\mathrm{P}_F(s)$$

$$t = 0 \to \mathrm{P}_O(0) = 1; \mathrm{P}_F(0) = 0$$

## Markov Model- Closed Form Solution

$$S\mathrm{P}_O(s) = 1-\lambda\,\mathrm{P}_O(s)+\mu\mathrm{P}_F(s)$$
$$S\,\mathrm{P}_F(s) = \lambda\,\mathrm{P}_O(s)-\mu\,\mathrm{P}_F(s)$$

$\rightarrow$

$$\mathrm{P}_O(s) = \frac{1}{s+(\lambda+\mu)} + \frac{\mu}{s(s+(\lambda+\mu))}$$
$$\mathrm{P}_F(s) = \frac{\lambda}{s(s+(\lambda+\mu))}$$

$$\mathrm{P}_O(s) = \frac{\frac{\mu}{\lambda+\mu}}{s} - \frac{\frac{\lambda}{\lambda+\mu}}{s+(\lambda+\mu)}$$
$$\mathrm{P}_F(s) = \frac{\frac{\lambda}{\lambda+\mu}}{s} - \frac{\frac{\lambda}{\lambda+\mu}}{s+(\lambda+\mu)}$$

$\rightarrow$

$$\mathrm{P}_O(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$$
$$\mathrm{P}_F(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

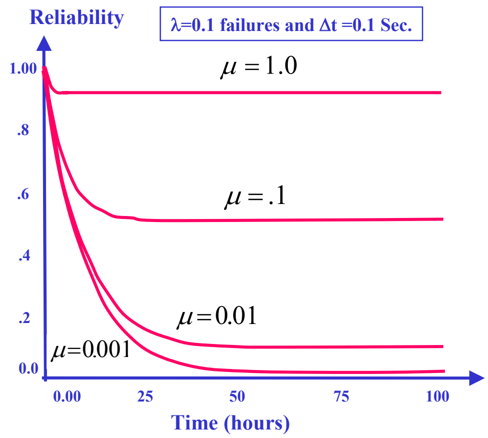
## Markov Model- Closed Form Solution

$$P_O(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

$$P_F(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

$$P_O(\infty) = \frac{\mu}{\lambda + \mu}$$

$$P_F(\infty) = \frac{\lambda}{\lambda + \mu}$$




## Steady State Availability

Availability A(t) of a system is the probability that the system will be operational at time t.

$$A(t) = \frac{\sum t_{Operationd}}{\sum t_{Operationd} + t_{repair}}$$

Good for experimental evaluation.
If not possible due to time and expense, we can estimate steady state availability

$$A_{SS} = \frac{MTTF}{MTTF + MTTR}$$

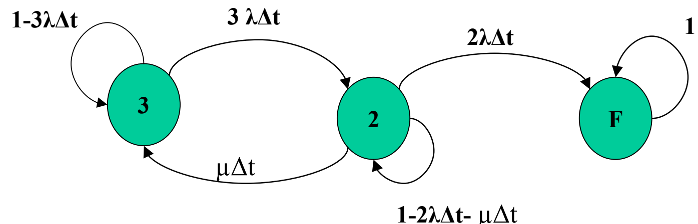
$$\text{MTTF} = \frac{1}{\lambda} \quad \text{MTTR} = \frac{1}{\mu}$$

$$A_{SS} = \frac{\mu}{\lambda + \mu} = P_O(\infty)$$

$$A_{SS} \rightarrow 1$$

$$if \quad \lambda = 0 \quad or \quad \mu = \infty$$

## Markov Model- TMR with Repair




- State matrix

$$\begin{bmatrix} P_3(t+\Delta t) \\ P_2(t+\Delta t) \\ P_F(t+\Delta t) \end{bmatrix} = \begin{bmatrix} (1-3\lambda\Delta t) & \mu\Delta t & 0 \\ 3\lambda\Delta t & (1-2\lambda\Delta t - \mu\Delta t) & 0 \\ 0 & 2\lambda\Delta t & 1 \end{bmatrix} \begin{bmatrix} P_3(t) \\ P_2(t) \\ P_F(t) \end{bmatrix}$$

- What if repair is allowed following the failure?