# Solutions
# Chapter 4

## Problem 2

$$\hat{H}\Psi_+(x,t) = \hat{H}\Psi_E(x,t) + \hat{H}\Psi_{E'}(x,t) = E\Psi_E(x,t) + E'\Psi_{E'}(x,t).$$

If $E \neq E'$ then the above result is not proportional to $\Psi_+(x,t)$. Hence, $\Psi_+(x,t)$ is not an eigenfunction of energy. However,

$$E\Psi_E(x,t) + E'\Psi_{E'}(x,t) = \hat{E}\Psi_E(x,t) + \hat{E}\Psi_{E'}(x,t) = \hat{E}(\Psi_E(x,t) + \Psi_{E'}(x,t)) = \hat{E}\Psi_+(x,t).$$

Hence,

$$\hat{H}\Psi_+(x,t) = \hat{E}\Psi_+(x,t)$$

which is the time dependent Schrödinger equation. Hence $\Psi_+(x,t)$ satisfies the time dependent Schrödinger equation.

## Problem 3

For $\psi_E = 0$, we need

$$\sin(n\pi x/L) = 0.$$

Hence,

$$x = mL/n, \quad m = 0, 1, 2, \ldots$$

As long as $0 \leq m \leq n$, $x$ is within the box. Hence, $m$ can take $n+1$ possible values (including boundary points).

## Problem 4

$$\begin{aligned}\int_0^L \psi_E(x)\psi_{E'}(x)dx &= (2/L)\int_0^L \sin(n\pi x/L)\sin(n'\pi x/L)dx \\ &= L^{-1}\int_0^L [\cos((n-n')\pi x/L) - \cos((n+n')\pi x/L)]dx.\end{aligned}$$

The integrals of both terms vanish if $n \neq n'$ ($E \neq E'$). However, if $n = n'$ ($E = E'$) only the second term has a zero integral. The first term is $\cos(0) = 1$. Hence, the result for $E = E'$.

## Problem 5

At the boundary point $x = L$,

$$\psi_E(x) = \sqrt{2/L}\sin(\pi/2) = \sqrt{2/L} \neq 0.$$

This does not satisfy the boundary condition.

## Problem 6

$$\Delta E = \frac{\pi^2\hbar^2(n'^2 - n^2)}{2mL^2}$$

For $n = 1$ and $n' = 2$,

$$\Delta E = \frac{3\pi^2\hbar^2}{2mL^2} = \frac{3h^2}{8mL^2} = 1.8 \times 10^{-37}\ \text{J}$$

## Problem 7

$$\int_{-\infty}^{\infty} \psi_0(x)\psi_1(x)dx = \sqrt{2/\pi}\frac{m\omega_0}{\hbar}\int_{-\infty}^{\infty} xe^{m\omega_0 x^2/\hbar}dx$$

The above integrand is an odd function of $x$ and the integration limits are symmetric. Such an integral is always zero.

## Problem 9

As $V_0 \to \infty$, $\kappa \to \infty$. Hence,

$$T = \left[1 + \frac{V_0^2\sinh^2(\kappa L)}{4E(V_0 - E)}\right]^{-1} \simeq \left[1 + \frac{V_0\sinh^2(\kappa L)}{4E}\right]^{-1}$$

The quantity in the square brackets tends to infinity as $V_0$ tends to infinity (Note that for large positive $x$, $\sinh x = e^x/2$.). Hence, $T$ tends to zero.

## Problem 11

The Schrödinger equation for $x < 0$ is

$$\frac{d^2\psi_E(x)}{dx^2} + \frac{2mE}{\hbar^2}\psi_E(x) = 0$$

Hence, its solution is of the form

$$\psi_E(x) = Ae^{ikx} + Be^{-ikx}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

The Schrödinger equation for $x > 0$ is

$$\frac{d^2\psi_E(x)}{dx^2} + \frac{2m(E - V_0)}{\hbar^2}\psi_E(x) = 0$$

Hence, its solution is of the form

$$\psi_E(x) = Ce^{iKx} + De^{-iKx}$$

where

$$K = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

As the beam is incident from the left, the right side cannot have a wave going left. Hence, $D = 0$. Now, as the wavefunction must be continuous at $x = 0$,

$$A + B = C$$

Also, as the derivative of the wavefunction must be continuous at $x = 0$,

$$kA - kB = KC$$

The above two boundary condition equations can be written as follows.

$$\begin{aligned} 1 + b &= c \\ 1 - b &= Kc/k \end{aligned}$$

where $b = B/A$ and $c = C/A$. Solving these gives

$$\begin{aligned} b &= \frac{1 - K/k}{1 + K/k} \\ c &= \frac{2}{1 + K/k} \end{aligned}$$

The reflection coefficient is

$$R = |b|^2 = \left[\frac{1 - K/k}{1 + K/k}\right]^2$$

## Problem 12

For $E < V_0$, $K$ is imaginary. Hence, the solution for $x > 0$ is

$$\psi_E(x) = Ce^{-\kappa x} + De^{\kappa x}$$

where

$$\kappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

This is not a travelling wave solution. So the argument that the left going wave must be absent does not hold. However, the wavefunction cannot become infinite at positive infinity. Hence, $D = 0$. Now, the boundary condition equations are the same as the last problem except for the the fact that $K$ is replaced by $i\kappa$. So,

$$\begin{aligned} b &= \frac{1 - i\kappa/k}{1 + i\kappa/k} \\ c &= \frac{2}{1 + i\kappa/k} \end{aligned}$$

For this case, the magnitude square of $b$ is not just its square because $b$ is not real. Hence,

$$R = |b|^2 = b^* b = \left(\frac{1 + i\kappa/k}{1 - i\kappa/k}\right)\left(\frac{1 - i\kappa/k}{1 + i\kappa/k}\right) = 1$$

## Problem 13

For example, the following three states have the same energy (degenerate states).

$$(n_x = 2, n_y = 1, n_z = 1), \quad (n_x = 1, n_y = 2, n_z = 1) \quad (n_x = 1, n_y = 1, n_z = 2)$$

For $L_x = L_y = L_z = L$,

$$E = \frac{\pi^2 \hbar^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$$

The lowest energy is for $n_x = n_y = n_z = 1$. this energy is

$$E_1 = \frac{3\pi^2 \hbar^2}{2mL^2} = 1.8 \times 10^{-37} \text{ J}$$

The next lowest energy is for the degenerate states mentioned above. The energy is

$$E_2 = \frac{6\pi^2 \hbar^2}{2mL^2} = 3.6 \times 10^{-37} \text{ J}$$