Unit Vector Notation

1 Definitions

Figure 1 shows the definition of components of a vector $\vec{\mathbf{V}}$. The angles α, β and γ are the angles that the vector makes to the three coordinate axes x, y and z respectively. The dashed lines are perpendiculars drawn from the tip of the vector to the three coordinate axis. The components V_x, V_y and V_z of $\vec{\mathbf{V}}$ are defined to be the segments of the coordinate axes marked out by these perpendiculars as shown. Hence, using the definition of cosine for the three right-angle triangles, we find,

$$V_x = V\cos\alpha, \quad V_y = V\cos\beta, \quad V_z = V\cos\gamma,$$

where V (short for $|\vec{\mathbf{V}}|$) is the magnitude of the vector $\vec{\mathbf{V}}$. We also define **unit vectors** (vectors of magnitude one) along each of the three coordinate axes x, y and z to be $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ respectively (figure 2). The following three vectors in the three coordinate directions can now be defined.

$$\vec{\mathbf{V}}_x = V_x\hat{\mathbf{i}}, \quad \vec{\mathbf{V}}_y = V_y\hat{\mathbf{j}}, \quad \vec{\mathbf{V}}_z = V_z\hat{\mathbf{k}}.$$

Using the triangle rule for vector addition twice, this gives,

$$\vec{\mathbf{V}} = \vec{\mathbf{V}}_x + \vec{\mathbf{V}}_y + \vec{\mathbf{V}}_z = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}} + V_z\hat{\mathbf{k}}.$$

This is known as the **unit vector notation** of a vector.

If the vector is restricted to the x-y plane, then $\gamma = 90°$. This makes (figure 3),

$$V_z = 0, \quad \text{and} \quad \alpha + \beta = 90°.$$

Hence,

$$\cos\beta = \cos(90° - \alpha) = \sin\alpha.$$

Thus the components V_x and V_y can now be written as,

$$V_x = V\cos\alpha, \quad V_y = V\sin\alpha$$

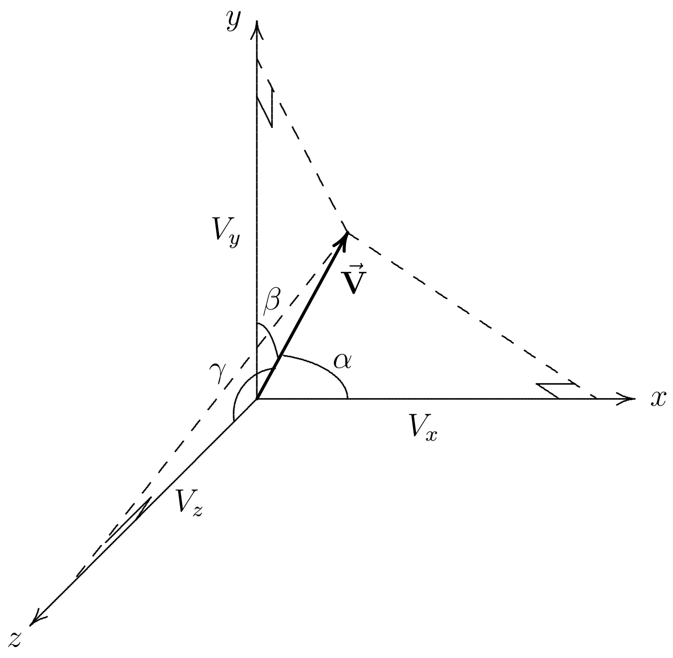

Figure 1: Components of a vector $\vec{\mathbf{V}}$

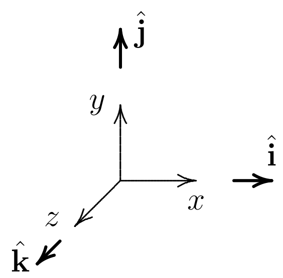

Figure 2: Unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ along the three coordinate directions.

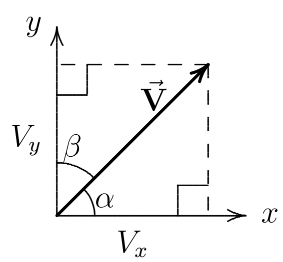

Figure 3: Components of a vector $\vec{\mathbf{V}}$ restricted to the x-y plane ($\beta = 90° - \alpha$).

2 The Dot (or Scalar) Product

The dot (or the scalar) product of any two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is written as $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ and defined to be

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = |\vec{\mathbf{A}}||\vec{\mathbf{B}}| \cos\theta = AB\cos\theta,$$

where θ is the angle between the two vectors and the notation $|\vec{\mathbf{A}}| = A$ is still being is used. The unit vector notation makes the computation of dot products rather easy. The unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are all perpendicular to each other and hence, from the above definition, their mutual dot products must be zero:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0.$$

Also from the above definition,

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1,$$

as the unit vectors have magnitudes of 1.

Now, if the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are given in unit vector notation, they are

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}, \quad \text{and} \quad \vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}.$$

Then their dot product is

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \cdot (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})$$

Using the distributive law of products, this gives 9 terms. However, using the dot products of the unit vectors as given above we get

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z.$$

3 The Cross Product

Unlike the scalar (or dot) product of two vectors, the cross product is a vector. Hence, its definition must include the direction as well as the magnitude.

- The **magnitude** of the cross product of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is given by

$$|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = AB\sin\theta, \tag{1}$$

  where A and B are the magnitudes of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ and θ is the angle between them (see figure 4).

- The **direction** of $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is defined to be perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. This means it is perpendicular to the plane containing the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Of course, this is ambiguous as two opposite directions are perpendicular to that plane. To resolve the ambiguity, the

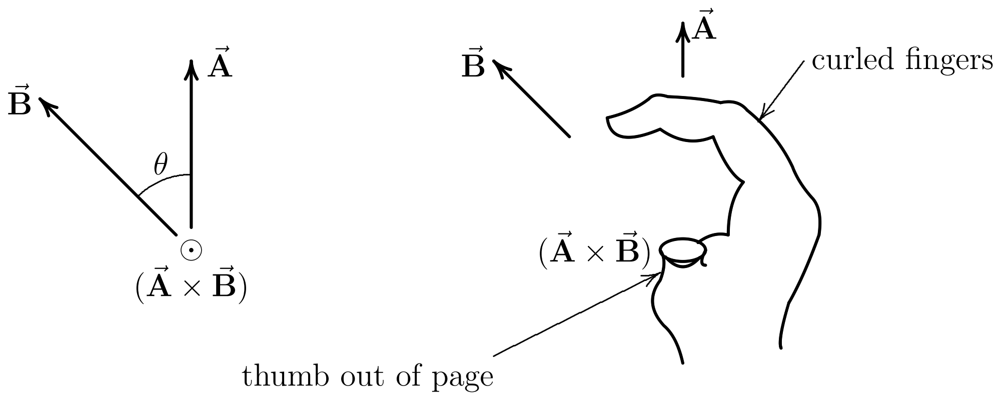

Figure 4: Finding the direction of the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ using the right-hand-rule. Note that the symbol $\odot$ means out of the page and $\otimes$ means into the page.

definition uses the so-called **right-hand-rule**. If you put the four fingers of the right hand together and curl them along the angle θ going from the first vector of the product ($\vec{\mathbf{A}}$ in this case) towards the second vector of the product ($\vec{\mathbf{B}}$ in this case), the direction in which the thumb will stick out is the direction of the cross product (see figure 4).

To compute cross products of vectors given in a unit vector notation, it is useful to know the cross products of the individual unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. From the above definition, it is straightforward to see the following.

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = -\hat{\mathbf{j}} \times \hat{\mathbf{i}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = -\hat{\mathbf{k}} \times \hat{\mathbf{j}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = -\hat{\mathbf{i}} \times \hat{\mathbf{k}} = \hat{\mathbf{j}}. \tag{2}$$

Also

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0. \tag{3}$$

Note that the definition of the cross product requires a minimum of three dimensions. So, drawing figures on a two-dimensional page becomes tricky. Hence, we shall often use a convention for representing directions out of the page and into the page. The symbol $\odot$ means out of the page and the symbol $\otimes$ means into the page. To remember these symbols, it is convenient to think of them as the tip (point) and the tail (feathers) of an actual arrow.