Solutions
Chapter 7

Problem 1

Part a

$$C = \frac{\epsilon_0 A}{d} = \frac{8.85 \times 10^{-12} \times \pi \times (0.10)^2}{1.0 \times 10^{-3}} = 2.8 \times 10^{-10} \text{ F}$$

Part b

$$q = CV = 1.4 \times 10^{-8} \text{ C}$$

Problem 2

If the equivalent of C_2 and C_3 is C_s, then

$$\frac{1}{C_s} = \frac{1}{C_2} + \frac{1}{C_3}$$

Hence,

$$C_s = \frac{C_2 C_3}{C_2 + C_3} = 10/3 \ \mu\text{F}$$

If the combination of C_s and C_4 is C_p then

$$C_p = C_4 + C_s = 4 + 10/3 = 22/3 \ \mu\text{F}$$

Finally, the net capacitance C is given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_p}$$

Hence,

$$C = \frac{C_1 C_p}{C_1 + C_p} = 5.37 \ \mu\text{F}$$

The charge in C_1 is the net charge in the combination C:

$$q_1 = q = CV = 5.37 \times 10 = 53.7 \ \mu\text{C}.$$

The charge in the combination C_p is the same (as it is in series with C_1). So the potential difference across C_p is

$$V_p = \frac{q}{C_p} = 7.32 \text{ V}$$

Hence, the charge in C_4 is

$$q_4 = C_4 V_p = 29.3 \ \mu\text{C}$$

The charge in the combination C_s is the charge in each of C_2 and C_3 as they are in series. Hence,

$$q_2 = q_3 = q_s = C_s V_p = 24.4 \ \mu\text{C}$$

Problem 3

Using the formula for parallel plate capacitors,

$$C_a = \frac{\epsilon_0 A}{a}$$

and

$$C_b = \frac{\epsilon_0 A}{b}$$

Hence, the equivalent capacitance C is given by

$$\frac{1}{C} = \frac{1}{C_a} + \frac{1}{C_b} = \frac{a}{\epsilon_0 A} + \frac{b}{\epsilon_0 A} = \frac{a+b}{\epsilon_0 A}.$$

Hence,

$$C = \frac{\epsilon_0 A}{a+b}.$$

Problem 4

The initial charge on capacitor C_1 is:

$$q = C_1 V = 5.00 \times 10^{-6} \times 20.0 = 1.00 \times 10^{-4} \text{ C}$$

When the second capacitor is connected, the charge q redistributes in the two capacitors as q_1 and q_2 such that: $q_1 + q_2 = q$. The final potential difference across each capacitor is $V' = 10.0$ V. Then

$$C_1 V' = q_1, \quad C_2 V' = q_2$$

which gives:

$$C_1 V' + C_2 V' = q_1 + q_2 = q$$

or

$$(C_1 + C_2) V' = q$$

Hence,

$$C_2 = \frac{q}{V'} - C_1 = \frac{1.00 \times 10^{-4}}{10.0} - 5.00 \times 10^{-6} = 5.00 \times 10^{-6} \text{ F}$$

Problem 5

Part a

The initial charges on the two capacitors are:

$$q_1 = C_1 V = 1.00 \times 50.0 = 50.0\ \mu\text{C}, \quad q_2 = C_2 V = 2.00 \times 50.0 = 100\ \mu\text{C}$$

When connected in reverse, the plate with charge $+q_1$ is connected to the plate with charge $-q_2$. So, together they share a total charge of $(q_1 - q_2)$. Hence, if their individual final charges are q_1' and q_2', then

$$q_1' + q_2' = q_1 - q_2 = -50.0\ \mu\text{C}$$

As the final potential difference V' across both capacitors is the same,

$$q_1' = C_1 V' \text{ and } q_2' = C_2 V'$$

Hence,

$$C_1 V' + C_2 V' = -50.0\ \mu\text{C}.$$

Solving for V' gives

$$V' = \frac{-50.0}{C_1 + C_2} = -16.7\ \text{V}.$$

The negative sign means that the plate of C_1 that was initially positive is going to end up as negative.

Part b

The charge on the first capacitor is

$$q_1' = C_1 V' = -16.7\ \mu\text{C}.$$

The charge on the second capacitor is

$$q_2' = C_2 V' = -33.3\ \mu\text{C}.$$

Problem 6

The quantity in question is the energy density which is

$$u = \epsilon_0 E^2/2 = 9.96 \times 10^{-8}\ \text{J/m}^3$$

Problem 7

Part a

If the initial charge on the first capacitor is q, then

$$q = CV.$$

Let C_p be the capacitance of the combination of the two capacitors. After the two capacitors are connected the voltage across the combination is $V/2$ and the total charge is still q. Hence,

$$q = C_p V/2$$

This gives

$$C_p = 2q/V = 2CV/V = 2C.$$

As C_p is the sum of the two capacitances, the second capacitor must have a capacitance of C as well.

Part b

The final voltage across each capacitor is $V/2$ and their capacitances are the same (C). Hence, the charge in each will be the same:

$$q_f = CV/2.$$

Part c

The initial energy in the first capacitor is

$$U_i = qV/2 = 10.0 \text{ J}.$$

After being connected, the total charge in the combination is still q. However, the potential difference is $V/2$. Hence the final energy in the combination is

$$U_f = q(V/2)/2 = U_i/2 = 5.00 \text{ J}$$

Part d

In the process of sharing the charge, there is a flow of charges through the wires. This flow (current) heats the wires. Hence, the other 5.00 J is lost as heat.

Problem 8

A charge density σ on the plate produces an electric field $E_0 = \sigma/\epsilon_0$ in the absence of any dielectric. A dielectric of dielectric constant κ_1 changes this field to $E_1 = E_0/\kappa_1$. Similarly, a dielectric of dielectric constant κ_2 changes this field to $E_2 = E_0/\kappa_2$. The total potential difference between the plates is the sum of the potential differences across each dielectric. Hence,

$$V = V_1 + V_2 = E_1 d/2 + E_2 d/2 = \frac{E_0 d}{2}\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right) = \frac{\sigma d}{2\epsilon_0}\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)$$

As $\sigma = q/A$,

$$V = \frac{qd}{2\epsilon_0 A}\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)$$

Hence,

$$C = q/V = \frac{\epsilon_0 A}{d}\left(\frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right).$$

Problem 9

Part a

The charge in a discharging capacitor decreases as follows.

$$q = q_0 e^{-t/\tau}$$

where the symbols have their usual meanings. If the first one-quarter of the charge is lost then the remaining charge is $q = 3q_0/4$. If the time taken for this is t_1, then

$$\frac{3q_0}{4} = q_0 e^{-t_1/\tau}$$

So,

$$\frac{3}{4} = e^{-t_1/\tau}$$

or

$$-\frac{t_1}{\tau} = \ln\left(\frac{3}{4}\right)$$

Hence,

$$t_1 = -\tau \ln\left(\frac{3}{4}\right) = 0.288\tau$$

Part b

If three-quarters of the initial charge is lost then $q = q_0/4$. If this happens in time t_2 then

$$\frac{q_0}{4} = q_0 e^{-t_2/\tau}$$

So,

$$\frac{1}{4} = e^{-t_2/\tau}$$

Hence,

$$t_2 = -\tau \ln\left(\frac{1}{4}\right) = 1.39\tau$$

Problem 10

Part a

Power delivered by the battery is

$$P = \mathcal{E} i = \mathcal{E}\frac{dq}{dt} = \mathcal{E}\frac{d}{dt}\left[C\mathcal{E}\left(1 - e^{-t/\tau}\right)\right] = C\mathcal{E}^2\frac{e^{-t/\tau}}{\tau} = \frac{\mathcal{E}^2}{R}e^{-t/\tau}$$

as $\tau = RC$. So the total energy delivered by the battery after a long time has elapsed is

$$U = \int_0^\infty P\, dt = \frac{\mathcal{E}^2}{R}\int_0^\infty e^{-t/\tau} dt = \frac{\mathcal{E}^2}{R}\tau = C\mathcal{E}^2$$

Part b

After a long time has elapsed, the current in the circuit is zero. So, there is no voltage drop across the resistor and the complete emf of the battery drops across the capacitor. Hence, the energy stored in the capacitor is

$$U_C = CV^2/2 = C\mathcal{E}^2/2.$$

This is only half the energy delivered by the battery.

Part c

The current in the circuit at any time t is

$$i = \frac{dq}{dt} = \frac{d}{dt}\left[C\mathcal{E}\left(1 - e^{-t/\tau}\right)\right] = \frac{\mathcal{E}}{R}e^{-t/\tau}$$

So the total heat loss is

$$U_T = \int_0^\infty i^2 R\, dt = R \int_0^\infty \frac{\mathcal{E}^2}{R^2} e^{-2t/\tau} dt = \frac{\mathcal{E}^2}{R} \frac{\tau}{2} = \frac{1}{2} C\mathcal{E}^2.$$

So, half the energy delivered by the battery is lost as heat through the resistor.